

# Visualizing some ideas about Gödel-type rotating universes.

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Some kinds of physical theories describe what our universe looks like. Other kinds of physical theories describe instead what the universe could be like independently of the properties of the actual universe. This second kind aims for the “basic laws of physics” in some sense which we will not make precise here (but cf. e.g. Malament [25, pp.98-99]). The present paper belongs to the second kind. Moreover, it is even more abstract than this, namely it aims for visualizing or grasping some mathematical or logical aspects of what the universe could be like.

The first six pages of this material are of a “science-popularizing” character in the sense that first we recall a space-time diagram from Hawking-Ellis [18] as “God-given truth”, i.e. we do not explain why the reader should believe that diagram. Then we derive carefully in an easily understandable visual manner an exciting, exotic consequence of that diagram: time-travel. This applies to the first six pages. The rest of this work is of a more ambitious character. *The reader does not have to believe anything*<sup>1</sup>. We do our best to make the paper self-contained and explain and visualize most of what we say.

In more detail, this work consists of Sections 1-8. Section 1 (p.2) is the just mentioned “popular” part. Section 2 (p.8) lays the foundation for discussing rotating universes. E.g. it shows how to visualize such space-times. The space-time built up in this section is called the “Naive Spiral world”. Section 3 (p.19) is about non-existence of a natural “now” in Gödel’s universe GU. Section 4 (p.22) introduces co-rotating coordinates “transforming the rotation away”. Section 5 (p.29) refines the Gödel-type universe (obtained in Section 2). Section 6 (p.46) illustrates a fuller view of the refined version of GU. Section 7 (p.52) re-coordinatizes the refined GU in order that the so-called gyroscopes do not rotate in this coordinatization. Section 8 (p.67) gives connections with the literature. E.g. it presents detailed computational comparison with the space-time metric in Gödel’s papers. Section 9 (p.70) contains technical data about how we constructed the figures illustrating Gödel’s universe.

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<sup>1</sup>Not even the diagram recalled from Hawking-Ellis [18] in Figure 1 or any of the statements made in the

1 Prelude: Some facts from the literature and how they imply time-travel.

The following series of figures represent Gödel’s famous rotating universe. One of the many interesting features of Gödel’s universe is that it contains closed time-like curves (CTC’s for short), i.e. it permits “time-travel”. In the following figures we use geodesics and light-cones in the spirit of e.g. [1, sections 3.1-3.3] for visualizing Gödel’s universe together with some of its main features. For these notions cf. p.8 herein. In Figures 1,2 null-geodesic is the same as photon-like geodesic and “null-cone” is the same as light-cone in the present paper.

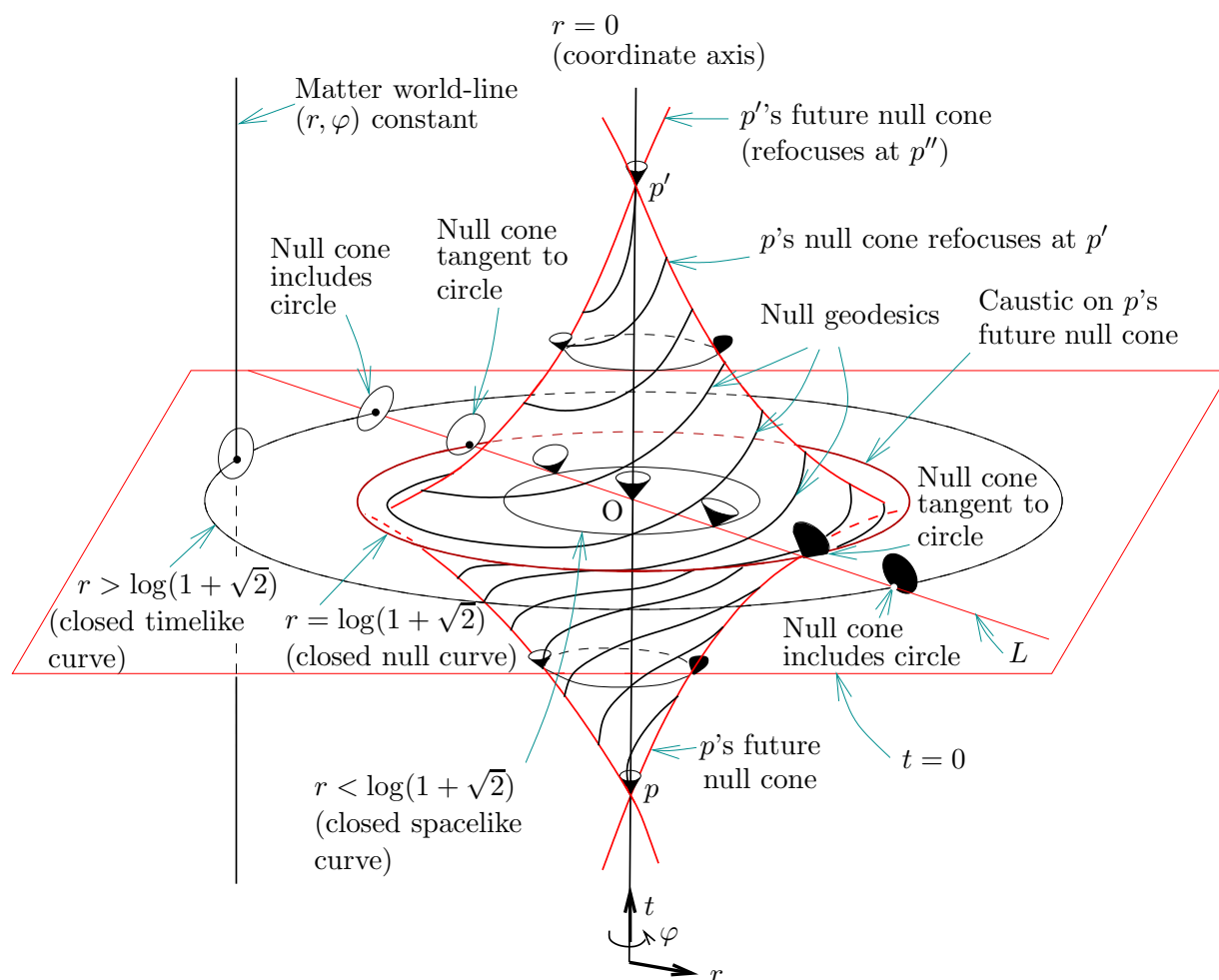


Figure 1: Gödel’s universe in co-rotating cylindric-polar coordinates  $\langle t, r, \varphi \rangle$ . Irrelevant coordinate  $z$  suppressed. Light-cones (null-cones) and photon-geodesics indicated. Light-cone *opens up* and tips over as  $r$  increases (see line  $L$ ) resulting in closed time-like curves (CTC’s). Drag effect (of rotation) illustrated. Photons emitted at  $p$  spiral out, reach CTC and reconverge at  $p'$ . This is a slightly corrected version of Figure 31 in Hawking-Ellis [18, p.169] (cf. p.69). (null cone = light-cone, null curve = photon curve)

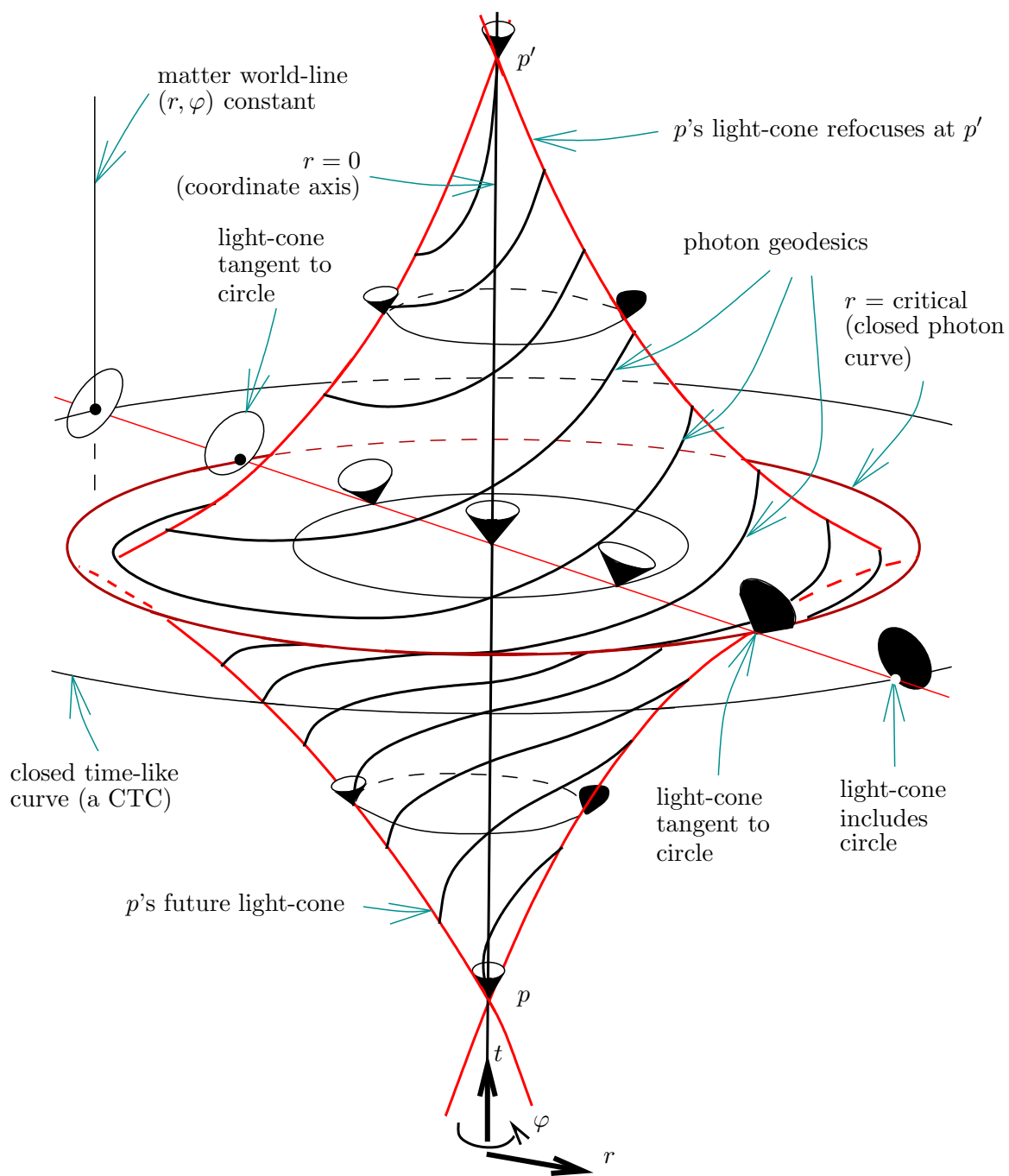


Figure 2: A closer look at Gödel's universe.

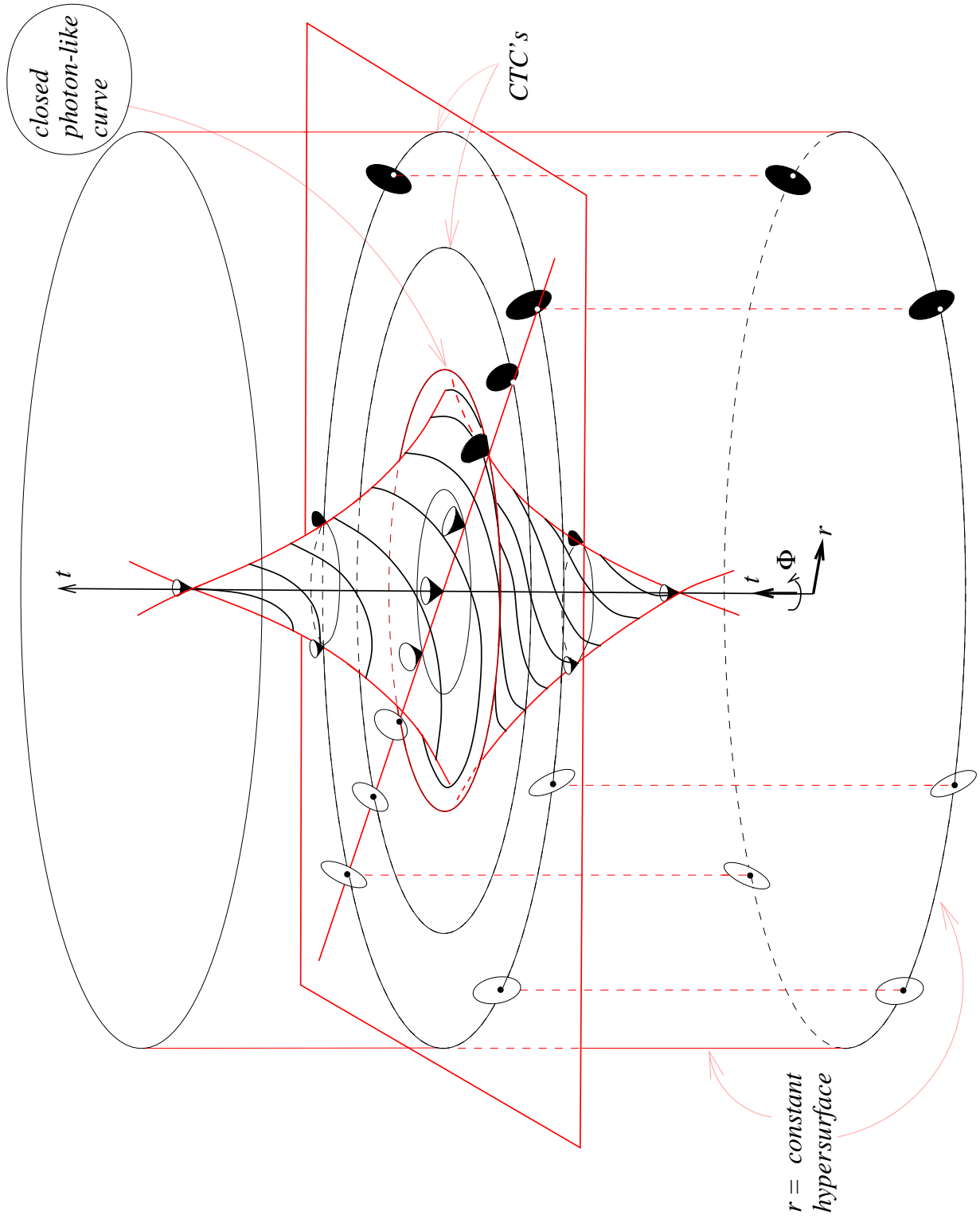


Figure 3: Gödel's universe as on previous figure but with an “ $r=\text{constant}$ ” (and  $z=\text{constant}$ ) hypersurface indicated. This hypersurface is parallel with the  $t$ -axis. Throughout this work,  $z=\text{constant}$ . I.e. throughout we suppress the irrelevant spatial coordinate  $z$ . In Figures 3-5,  $\Phi$  is the same as  $\varphi$  in the rest of the paper.



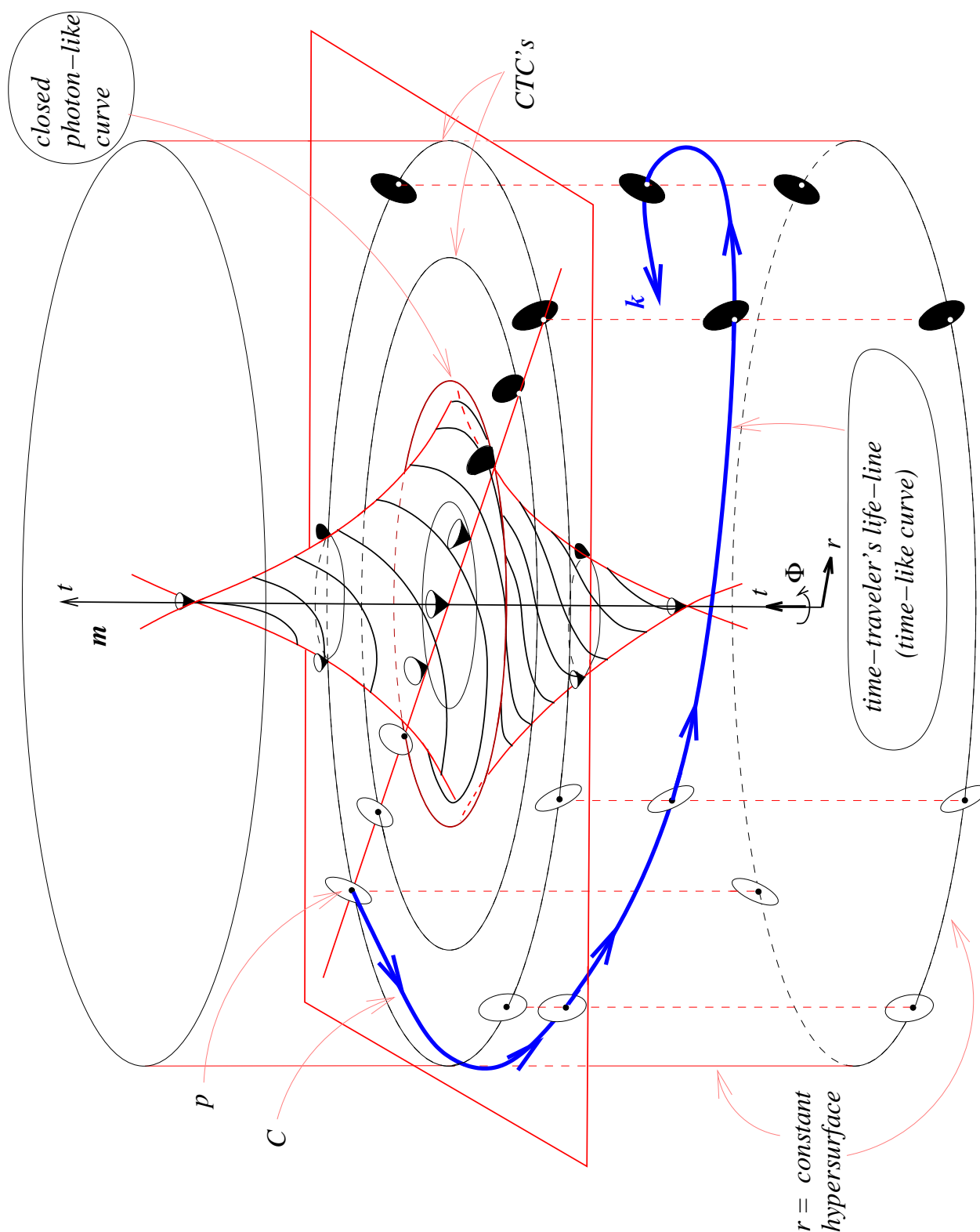


Figure 4: Gödel’s universe with a time-traveler’s (time-like) life-line indicated. The time-traveler’s acceleration is bounded (but cannot be zero). The time-like curve  $C$  stays always inside the light-cones and spirals back to the past as  $m$  observes it. This is possible because the light-cones far away from the  $t$ -axis are so much tilted that they reach below the horizontal plane. See the explanation on p.7.

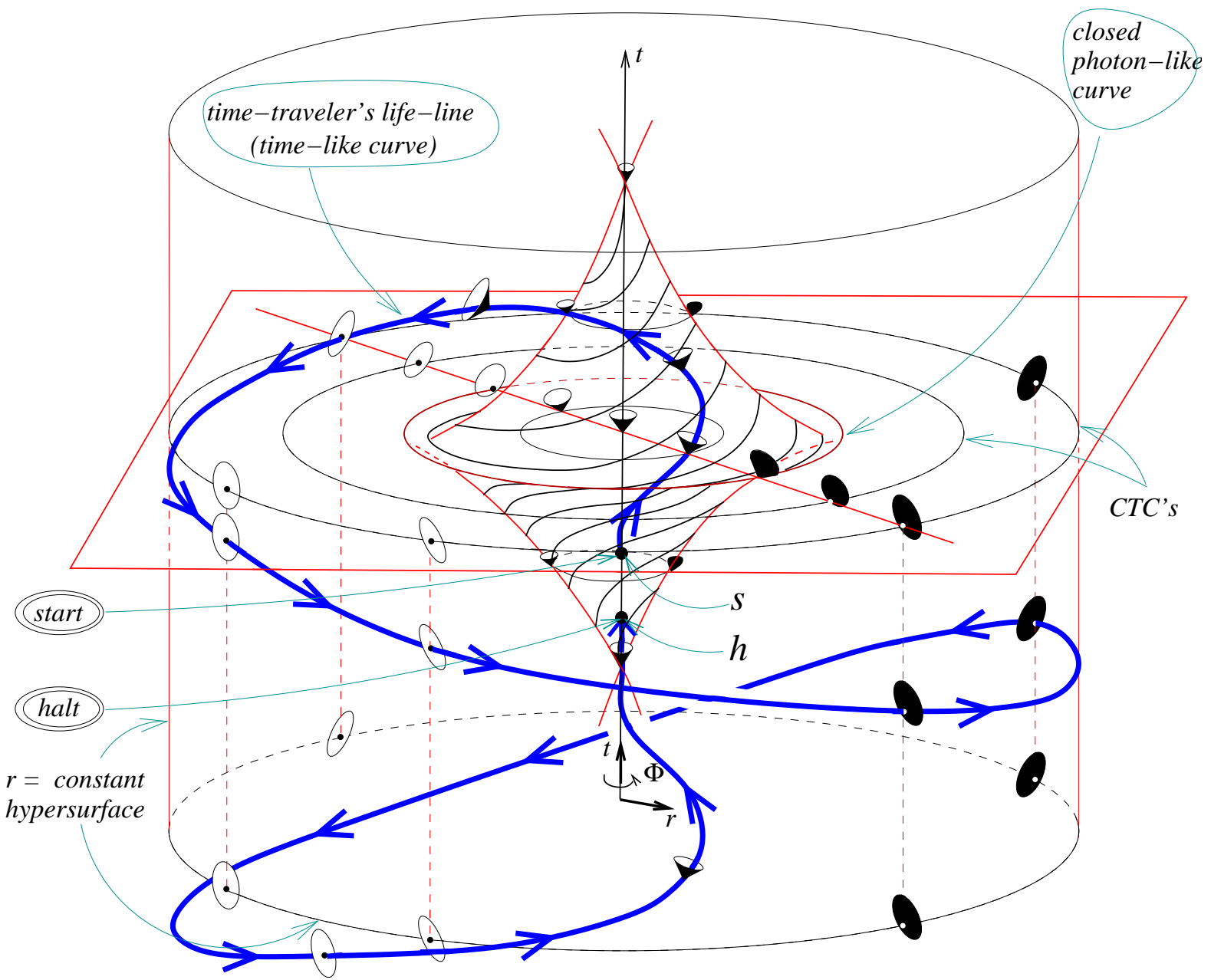


Figure 5: Time-traveler starting at time  $s$  and arriving at time  $h$ , where  $h$  is earlier than  $s$ .

Explanation for Figures 4,5: Figures 4,5 illustrate the time-travel aspect in Gödel's universe. Assume observer  $m$  lives on the time axis  $\bar{t}$ . Assume  $p$  is a point far enough from  $\bar{t}$ . I.e. the radius  $r$  of  $p$  is large enough. Then at  $p$  the light-cones are so much tilted that a time-like curve  $C$  can spiral back into the past as observed by  $m$ .  $C$  involves only bounded acceleration. An observer, say  $k$ , can live on  $C$ . Then in  $m$ 's view,  $k$  moves towards the past. Moreover,  $k$  can go back to the past as far as he wishes.

It is an entertaining exercise to prolong curve  $C$  such that it starts at  $s \in \bar{t}$  and ends at  $h \in \bar{t}$  such that  $h \prec s$ , i.e.  $h$  is in the past of  $s$ , see Figure 5. Then our observer  $k$  can start its journey at  $s$ , spiral outwards to radius  $r$ , then spiral back along  $C$  and then spiral inwards to  $h$ . Then  $k$  can wait on the time axis  $\bar{t}$  to meet itself at point  $s$ . We leave the details to the reader, but see Figure 5.

Cf. also Figure 28 on p.113 in Horwich [21], which we include below.

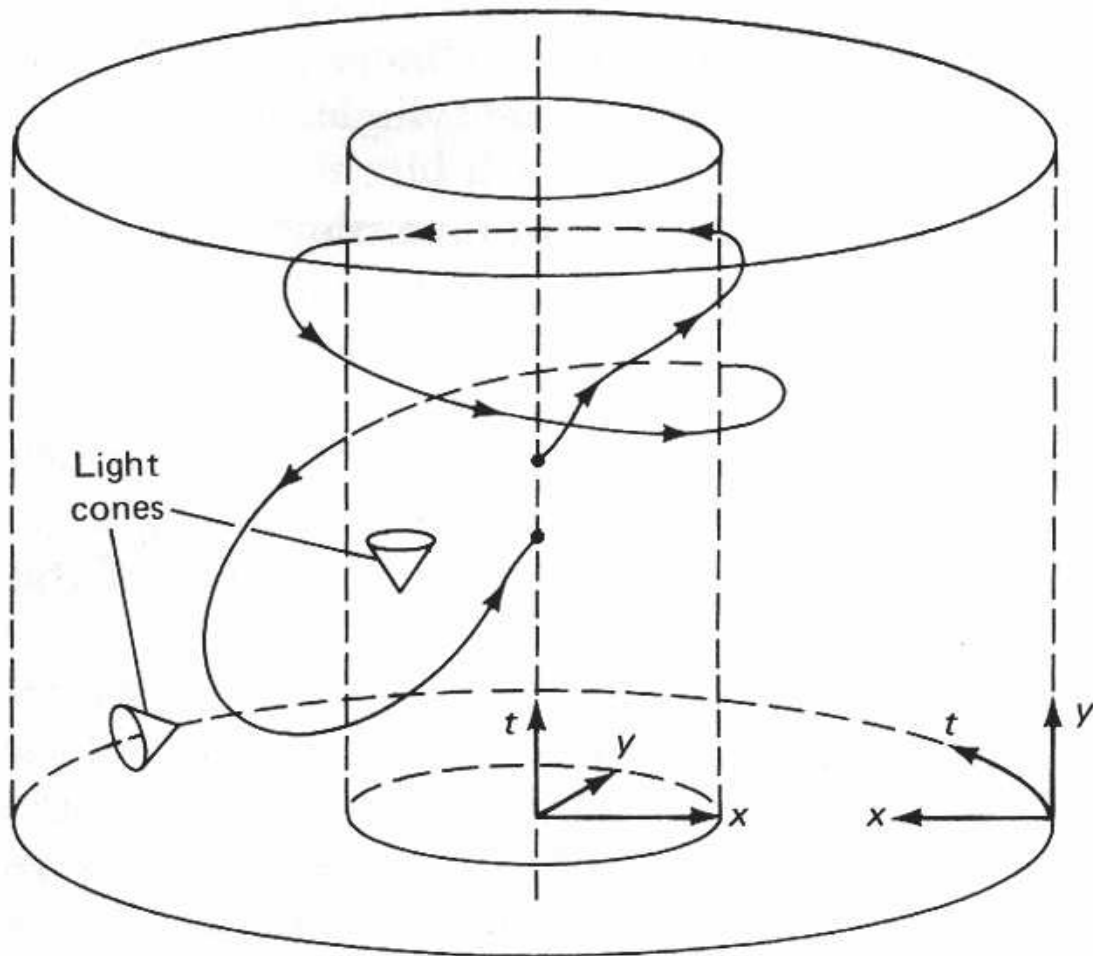


Figure 6: Figure from Horwich [21, Figure 28 (p.113)].

## 2 Preparation for constructing Gödel style rotating universes. The Naive Spiral World.

In this part we populate Newtonian space with massive observers  $m_i$  for  $i \in I$  which carry equal mass and are evenly distributed (where we understand “even” in the common sense). We will call these  $m_i$ ’s *distinguished observers* or *mass-carriers* or *galaxies*<sup>2</sup>. Then we rotate this inhabited space around the  $z$  axis. The galaxy in the origin is called  $m_0$ . We will make sure that nothing happens in the direction  $z$ , therefore we can suppress direction  $z$  in our pictures and discussion. So space-time becomes three-dimensional with axes  $t, x, y$ . We concentrate on the  $xy$ -plane inhabited by the galaxies (or distinguished observers)  $m_i$ . We rotate this plane of galaxies around the origin, i.e. around  $m_0$ . The rotation is rigid, i.e. the distances between the galaxies do not change. The *angular velocity* of this rotation is denoted by  $\omega$ . We call the plane inhabited by the  $m_i$ ’s the *universe*. Hence  $\omega$  is called the angular velocity of the universe. The rotation takes place in a Newtonian inertial frame of reference.<sup>3</sup> The angular velocity  $\omega$  is chosen such that the resulting centrifugal force exactly balances the gravitational attraction between the  $m_i$ ’s. This is possible, cf. Gödel’s paper [15, second half of p.270] for a proof. (Cf. [15, pp.261-289] for more detail.)

So our first pictures will show space-time diagrams in which the life-lines<sup>4</sup> of the galaxies  $m_i$  appear as spirals around the  $t$ -axis (which happens to be the life-line of  $m_0$ ). An extra feature is that, similarly to Gödel’s papers, we assume the existence of certain kinds of *cosmic compasses*. Our cosmic compasses need not agree with what are called gyroscopes in physics. For the time being cosmic compasses constitute only certain conventions. Equivalently, they can be regarded as distinguished *local coordinate frames* or “local coordinate systems” for our distinguished observers or mass-carriers (the  $m_i$ ’s). These local frames need not be inertial. For the time being we do not associate any tangible or observational physical meaning to our compasses and local frames.<sup>5</sup> In Section 7 we will turn our attention to gyroscopes and local inertial frames, too.

We assume that all the  $m_i$ ’s agree with each other in that they have two cosmic compasses for carrying the original spatial directions  $x$  and  $y$  of our original Newtonian inertial reference frame with which we began our construction. This makes them equivalent (with each other) in the sense that any of them, say  $m$ , may think that he is at the center, he is not rotating and it is the rest of the observers who are rotating around  $m$ .

This paper is based on general relativity but we do not assume that the reader is familiar with the details of general relativity. What we do assume is familiarity with (i) the basics of special relativity and (ii) awareness of some of the basic principles of general relativity explained in items (1)-(2) below. All this can be found in [1]. All what we need to know about special relativity in this paper can be found in [1, sections 2.1-2.4]. What we need to know

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<sup>2</sup>We use the word “galaxy” only in a metaphorical sense and it means nothing more than our distinguished observers carrying mass. Cf. Rindler [33, p.203] for more on our usage for galaxies.

<sup>3</sup>Here we use the expression “inertial frame of reference” in the most classical (Newtonian) way, namely as it was given by L. Lange in 1885: “A reference frame in which a mass point thrown from the same point in three different (non co-planar) directions follows rectilinear paths each time it is thrown, is called an inertial frame.”

<sup>4</sup>What we call life-line is called world-lines in most of the literature of general relativity.

<sup>5</sup>What they represent is mainly a logical “stage” in our construction of rotating universes. Though, in which they represent (if they represent) observational phenomena, then we have to insist that

about general relativity theory, summarized in items (1)-(2) below, can be found in [1, sections 3.1-3.3].

(1) General relativity assumes that *special relativity holds locally*. This means, roughly, that in a general relativistic space-time, every point (event) is “surrounded” by a small, local coordinate frame (LF for short) and in each LF special relativity holds in some sense (cf. e.g. Rindler [33] for a simple explanation of this). The LF’s are local in the topological sense that space-time  $M$  comes together with a topology and then LF’s are local in the sense that the “closer” we go to the point  $p \in M$  the more accurately the local special relativity frame LF describes the behavior of light-signals and moving bodies. (For a precise formulation see [1, sec.3.3, e.g., Def.3.3].)

In the case of Gödel’s universe,  $M$  together with this topology is just the original (Newtonian) space-time  $\mathbb{R}^4$ . Thus, in the case of Gödel’s universe  $\langle M, \dots \rangle$  a single “global” coordinate system can cover the whole of  $M$ . This means that there exist coordinatizations  $Co : \mathbb{R}^4 \longrightarrow M$  with  $Co$  a bijection which satisfy some natural requirements which we do not list here. E.g.  $Co$  involves one “time coordinate” and three “space coordinates”, hence at first glance it looks similar to the familiar coordinatization of Newtonian space-time or special relativity. Further, one of the space coordinates turns out to be irrelevant, hence  $Co : \mathbb{R}^4 \longrightarrow M$  will admit a 3-dimensional representation (via suppressing the irrelevant coordinate). So in our pictures there will be *one big coordinate system*  $Co$  covering the whole picture and there will be *many small coordinate systems* representing the LF’s or other local coordinate systems. The big coordinate system represents the whole of our manifold  $M$  to be described.

When we describe a space-time  $M$ , the key ingredient is specifying how the little LF’s are glued together to form the whole of  $M$ . We will do this by specifying a (fairly arbitrary) coordinatization  $C$  of  $M$  and then to each point  $p \in M$  we describe how the LF at  $p$  is fitted into  $M$  at point  $p$ .<sup>6</sup> When specifying which LF is glued to what point, we use the coordinate system  $C$  as a tool for communication. Most of the time we will use geometric constructions for presenting the above data. In such a picture, the LF at  $p$  is represented by drawing the *light-cone* at  $p$  together with the *unit vectors*  $\langle t_p, x_p, y_p \rangle$  of the LF at  $p$ . Sometimes we indicate only the future light-cones, sometimes we indicate both the future and the past light-cones. Most of the time we indicate the local simultaneity of the LF, too.<sup>7</sup> These pictures, beginning with Figure 12, represent *precise geometrical constructions*, hence they intend to specify the space-time in question completely (as opposed to being a mere “sketch” conveying only intuitive ideas). In Sections 9,8 which contain the technical details we present the constructions behind the pictures together with the metric tensor field of the space-time in question. (To explain the latter, we note that a model of general relativity is usually given in the form  $\langle M, \mathbf{g} \rangle$  where  $M$  is a manifold and  $\mathbf{g}$  is a tensor field defined on  $M$ . We will not need these tensor-fields until Section 8.) We note that  $\mathbf{g}$  can be reconstructed from the way the LF’s are glued together in our pictures, hence if the reader understands the geometry of these pictures, he will automatically understand the space-time (or general relativity model) they represent.

(2) Occasionally we will mention so-called *geodesics*. Geodesics are the general relativistic counterparts of straight lines of special relativity, in particular, the life-lines of inertial bodies or freely falling bodies are called geodesics. The same applies to life-lines of photons. *Curves* are understood in the usual sense, e.g. geodesics are special curves. Properties of curves are

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<sup>6</sup>The effect is somewhat similar to an Escher painting, e.g. he glues little birds together and there emerges an over-all pattern which has nothing to do with birds.

<sup>7</sup>To specify the LF, it is enough to specify the unit vectors  $\langle t_p, x_p, y_p \rangle$ . These determine the light-cones and the local simultaneity. However, the latter are very helpful in visualizing the space-time, that’s why we indicate them in the pictures.

generalized from special relativity to general relativity by saying that curve  $\ell$  has property  $P$  if it has  $P$  locally (in the sense of special relativity). E.g.  $\ell$  is *time-like* if for each  $p \in \ell$  the LF surrounding  $p$  “thinks” that  $\ell$  is time-like in the sense of special relativity. Similarly for *space-like*, *photon-like* (and for other properties of geodesics).

We note that time-like curves are the possible life-lines of arbitrary bodies, i.e. of not necessarily inertial bodies. These may undergo acceleration. Both geodesics and time-like curves are curves in the usual sense. A curve is time-like if it always stays inside the light-cones. A curve  $\ell$  is photon-like if for any point  $p \in \ell$ ,  $\ell$  is tangent to the light-cone at  $p$ .

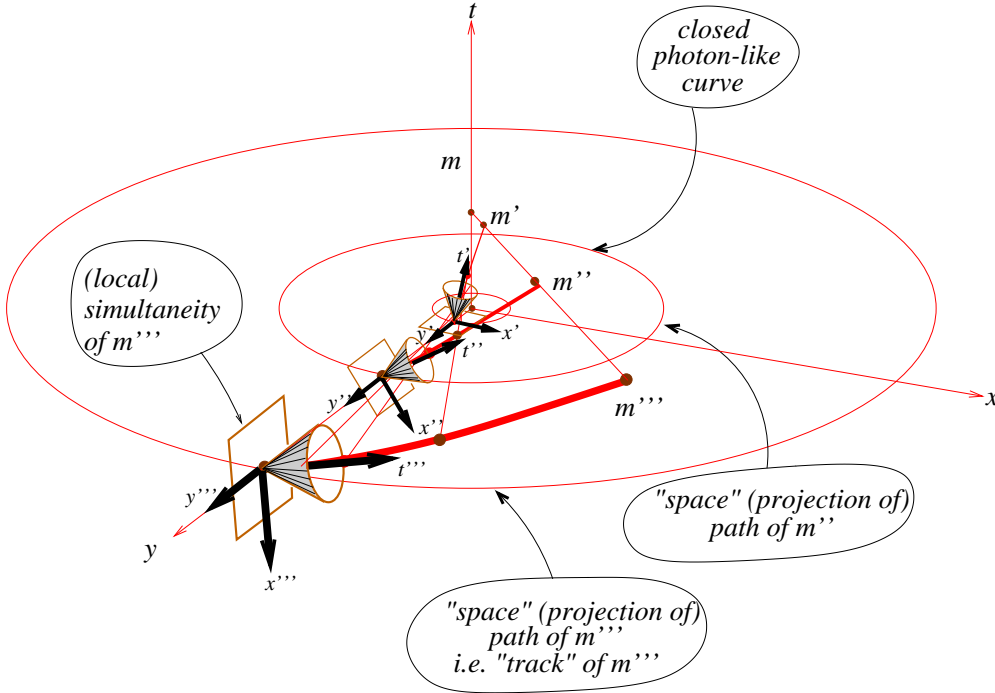


Figure 7: Observers  $m', m'', m'''$  perform a rigid rotation around observer  $m$ . Such observers are the only mass-carriers in this universe. Because of this rotation,  $m'''$  moves so fast that his light-cone tilts over so much that it is almost horizontal.

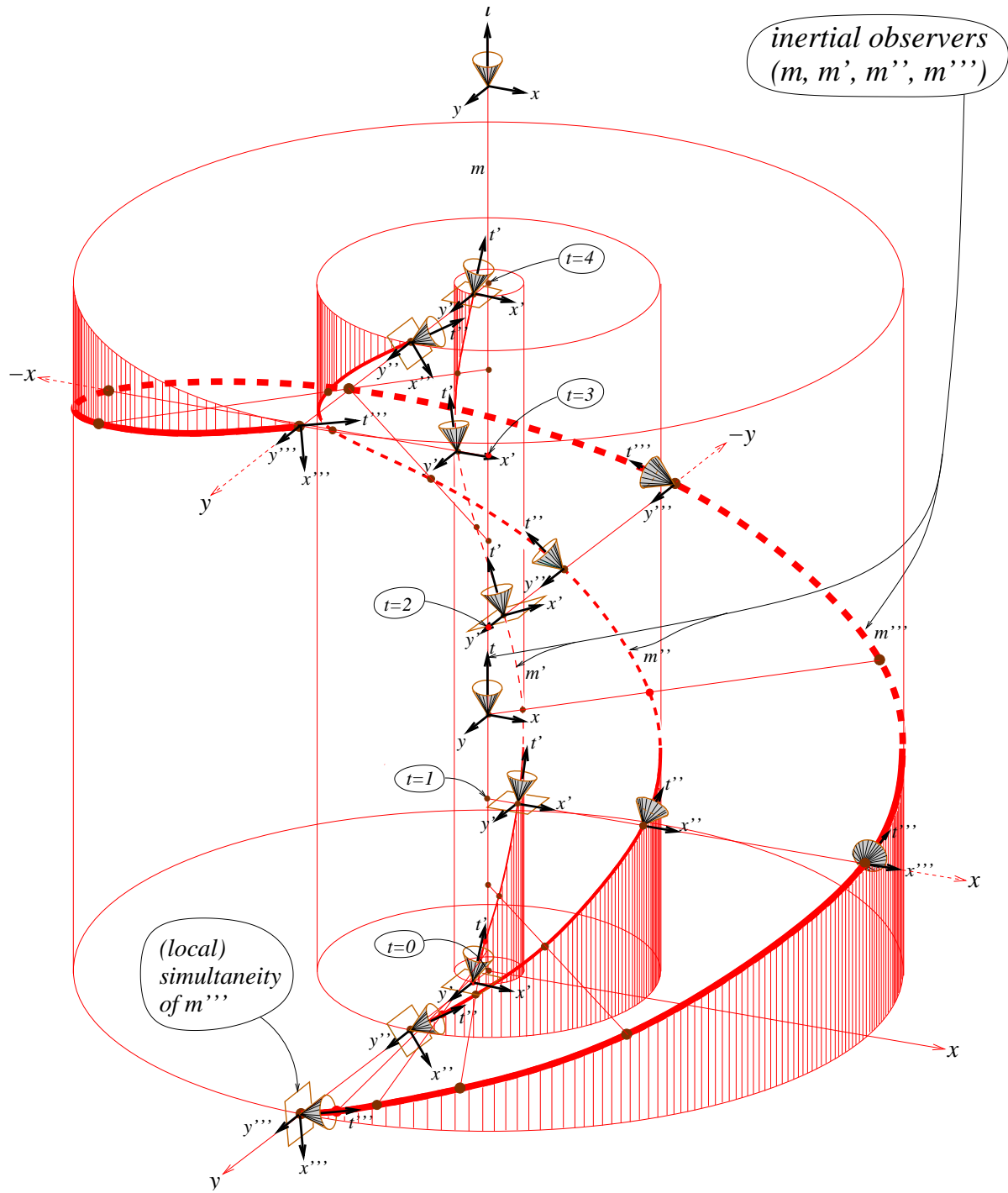


Figure 8: Gödel's Universe with emphasis on inertial observers instead of photons (the rotation is "rigid"). The coordinate system  $\langle t', x', y' \rangle$  of say  $m'$  does not follow the rotation of the matter in this universe. The life-lines of  $m, \dots, m'''$  are (special) geodesics.  $\langle t, x, y \rangle, \langle t', x', y' \rangle$  etc. are distinguished local coordinate systems. E.g.  $\langle t'', x'', y'' \rangle$  is the local coordinate system of observer  $m''$ .

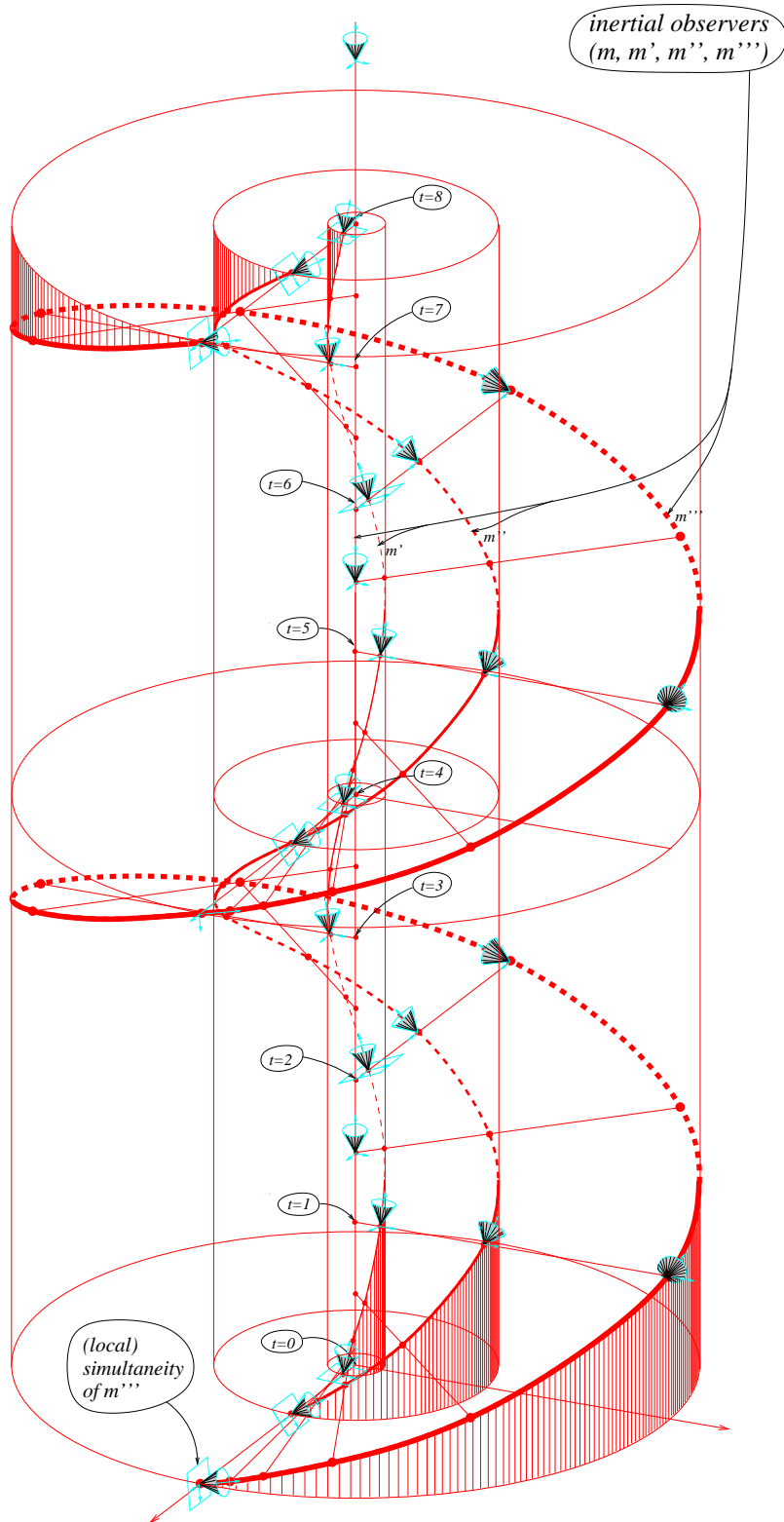


Figure 9: Previous figure copied on top of itself. It goes on like this in both directions forever.  $m', m'', m'''$  are (time-like life-lines of) observers “equivalent with” the observer  $m$  living on  $\bar{t}$ .



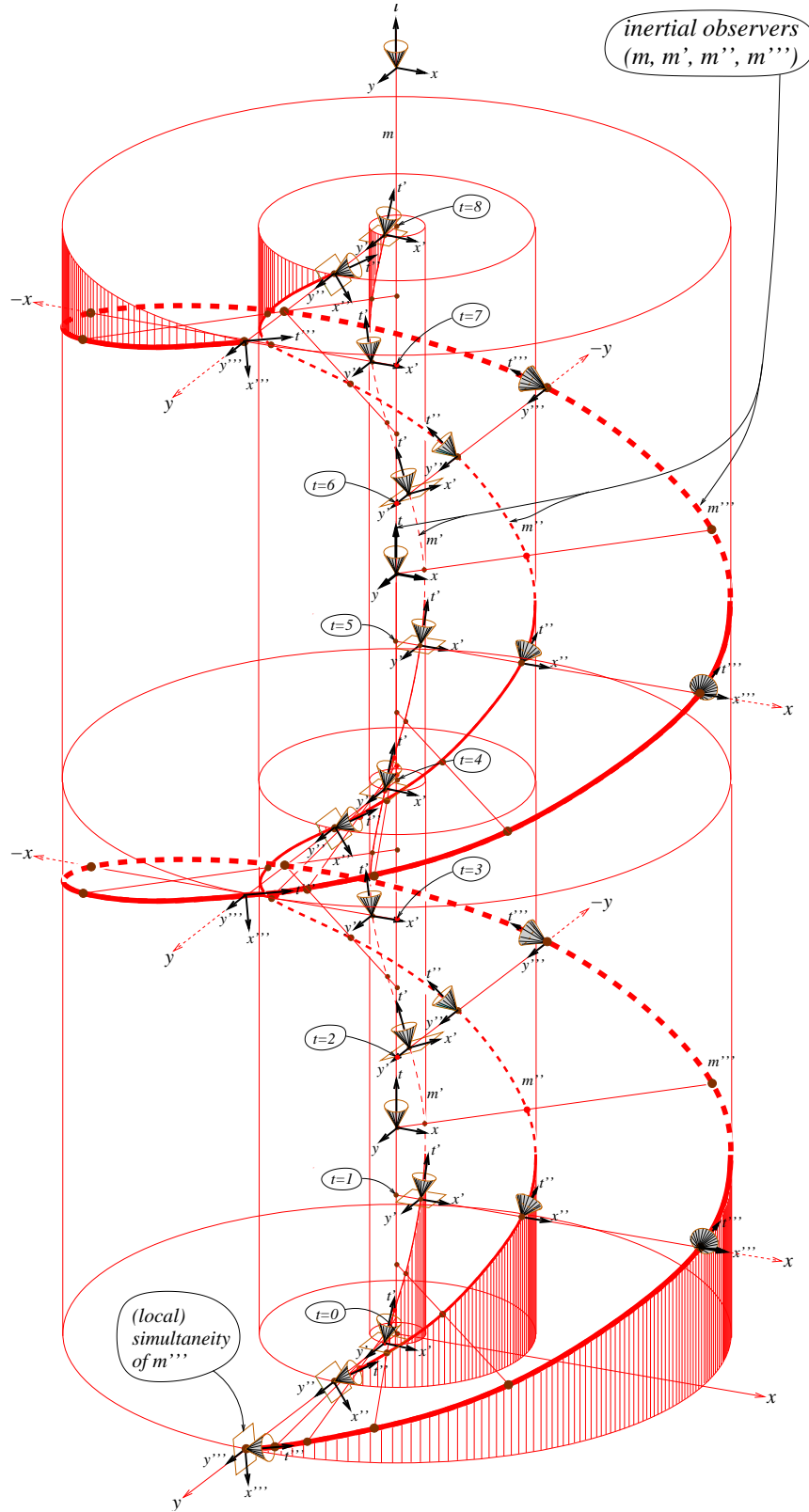


Figure 10: Previous figure with non-rotating local coordinate systems  $\langle t', x', y' \rangle$ ,  $\langle t'', x'', y'' \rangle$  etc. emphasized.

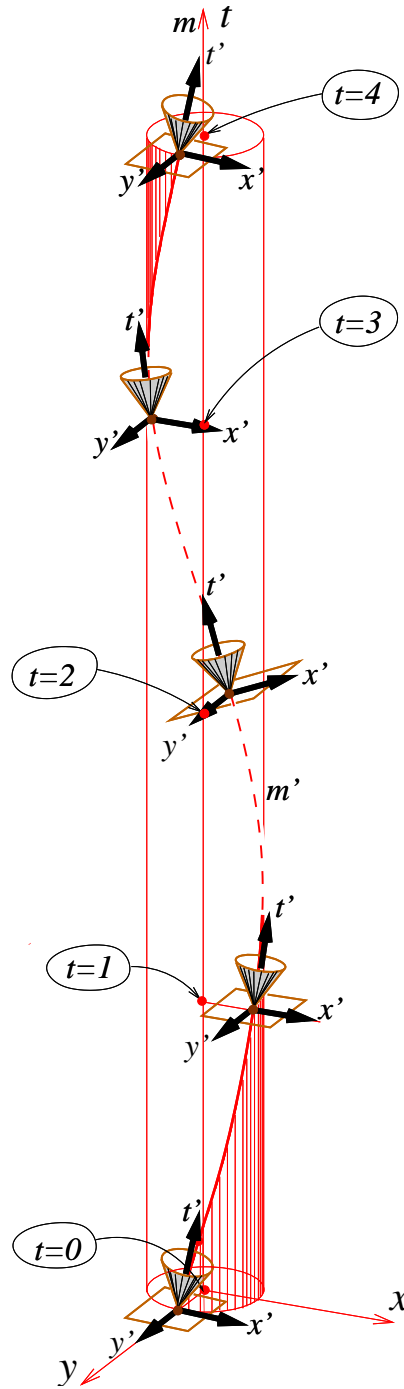


Figure 11: The coordinate system  $\langle t', x', y' \rangle$  of say  $m'$  does not follow the rotation of the matter in this universe. The reader is asked to check that in a certain sense the direction  $x'$  remains parallel with the original direction  $x$ . This is why  $m'$  thinks that  $m$  is rotating around  $m'$ .

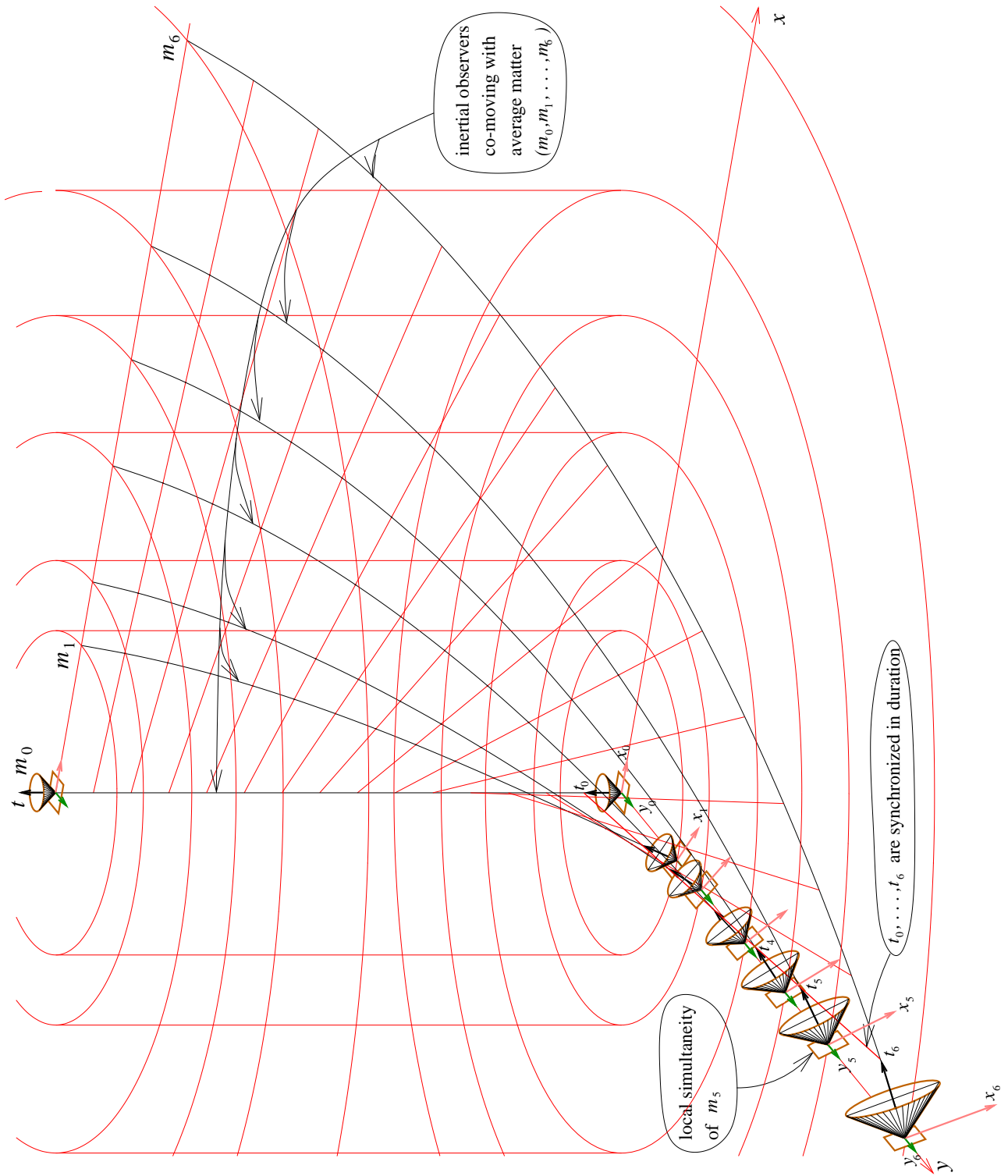


Figure 12: Each  $m_i$  can measure the time needed for a single turn of the universe. (I.e. each  $m_i$  can measure the angular velocity  $\omega$  of the universe.) To ensure this we have to calibrate the  $t_i$  vectors of the  $m_i$ 's such that in  $m_0$ 's view the vertical components of all the  $t_i$ 's are *equal* with that of  $t_0$ .  $\omega = \pi/30$ , Map 2 applies. Cf. p.81.

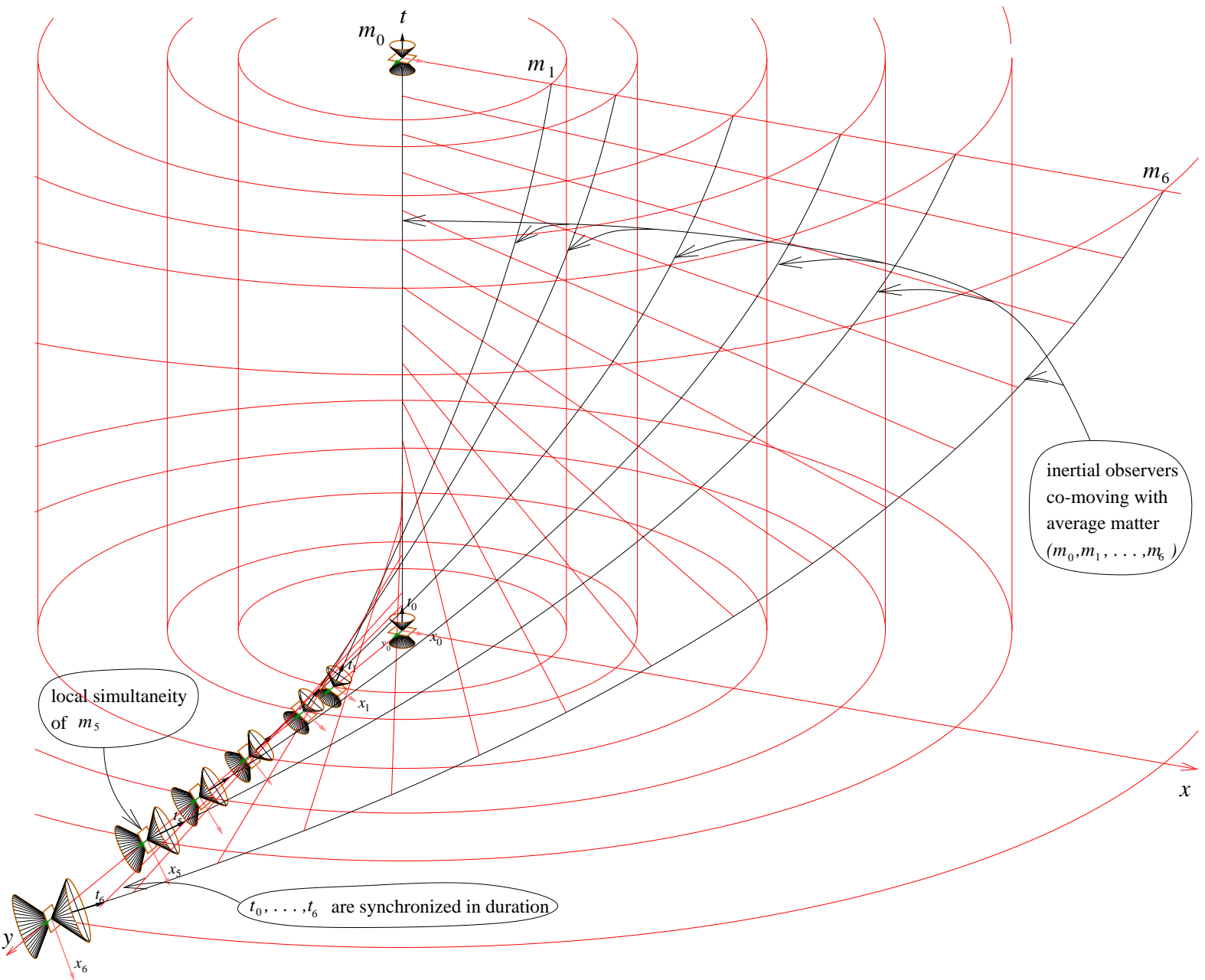


Figure 13: Previous figure with past-light-cones indicated.  $\omega = \pi/45$ , Map 1 applies. Cf. p.80.

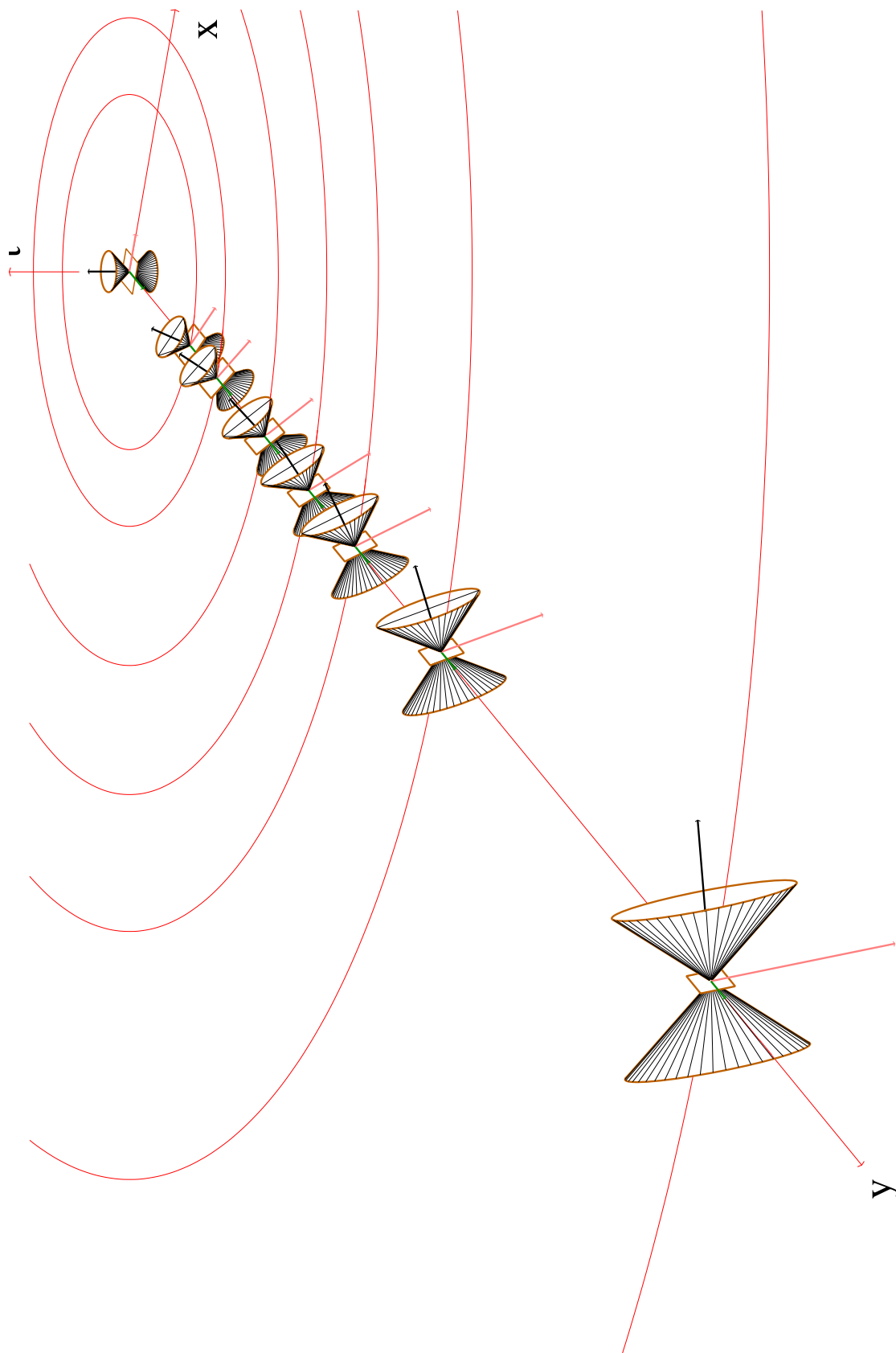


Figure 14:  $\omega = \pi/30$ , Map 2 applies. Cf. p.81.

Gödel wanted the distinguished massive observers  $m_0, \dots, m_i, \dots$  of his universe to be equivalent with each other. So far they are equivalent from the point of view that each of them thinks that the rest of the universe rotates around himself. This is so because the local coordinate systems (hence the cosmic compasses) of the distinguished observers  $m_i$  do not rotate, do not follow the rotation of the universe. At this point we can ensure one more symmetry property of the  $m_i$ 's. Each  $m_i$  can measure the time needed for a single turn of the universe, for example as follows:  $m_i$  picks a distinguished observer, say  $m_0$ , such that  $m_i$ 's  $y$ -compass points in the direction of  $m_0$  at an instant, and then measures the time passed until his  $y$ -compass again points in  $m_0$ 's direction.<sup>8</sup> This is how  $m_i$  can measure the angular velocity  $\omega$  of the universe. To ensure that all the distinguished observers get the same value for the angular velocity, we have to calibrate the  $t_i$  vectors of the  $m_i$ 's such that in  $m_0$ 's view the vertical components of all the  $t_i$ 's are *equal* with that of  $t_0$ . This is ensured in Figure 12, and from now on we will always ensure this.<sup>9</sup> This choice of the local time-unit vectors ensures also that the local LF's measure a kind of "universal time", namely that of the big global reference frame. However, this "universal time" does not satisfy natural requirements about "time" presented in the next section.

Above we specified the time-unit-vectors of the local frames. Let us now specify three other unit-vectors at each point  $p$ , these will specify the light-cone and the local special relativity at  $p$ . All what we say below in specifying the three unit vectors are meant in the big global reference frame. The  $r$ -unit-vector at  $p$  points in the radial direction parallel to the  $xy$ -plane and has length 1. The (suppressed)  $z$ -unit-vector points in the direction of the (suppressed)  $z$ -axis and has length 1. Finally, the last unit-vector is orthogonal to the three unit-vectors given so far and has the same length as the  $t$ -unit-vector. In the local frame at  $p$ , these 4 vectors constitute an orthonormal system. By this, we specified fully our general relativistic space-time.<sup>10</sup>

The *preliminary* version of Gödel's universe GU constructed above and depicted in Figures 7-14 will be referred to as "Naive GU" (NGU) or more specifically, "*Naive Spiral World*". The reason for this is that so far we have chosen the simplest possible arrangement of light-cones without checking whether they will satisfy certain properties we have in mind. Indeed, Section 5 will lead to some refinement/fine-tuning of the light-cone structure. However, the Naive GU has many of the desired properties already. Namely, the life-lines of the galaxies are geodesics, i.e., the distinguished observers  $m_i$  are really inertial observers. The radial straight lines parallel to the  $xy$ -plane are all geodesics, too.

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<sup>8</sup>What does it mean that  $m_i$ 's  $y$ -compass points in  $m_0$ 's direction at some time  $t$ ? We may use the following definition: there is a curve  $\ell$  connecting  $m_i$ 's life-line (starting with the event at  $t$ ) with  $m_0$ 's life-line such that at each point  $p$  of the curve  $\ell$  the following holds:  $\ell$  lies in the local simultaneity of the distinguished observer  $m$  passing through  $p$  and  $m$ 's  $y$ -compass points in  $\ell$ 's direction in  $p$ .

<sup>9</sup>This will also ensure that each  $m_i$  will measure the same angular velocity for the universe, no matter which "partner" he chooses (in place of  $m_0$ ) for the measurement.

<sup>10</sup>The metric tensor is  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ .

### 3 Non-existence of a global time in Gödel's universe.

Figures 15–16 below form an informal illustration for the idea of “non-foliasibility” of Gödel's universe GU. I.e. Figures 15–16 intend to illustrate the claim that there is no global natural simultaneity in GU.

By a *potential simultaneity* of GU we can understand a hyper-surface  $S$  in the usual sense and we can require it to satisfy conditions like (i)–(vi) below.

- (i)  $(\forall p, q \in S)[p \neq q \Rightarrow (\exists \text{ maximal space-like geodesic } \ell)(p, q \in \ell \subseteq S)]$ .
- (ii)  $(\forall \text{ space-like geodesic } \ell)[\text{a nonempty open segment of } \ell \text{ lies in } S \Rightarrow \ell \subseteq S]$ .
- (iii) Every maximal time-like geodesic  $\ell$  intersects  $S$  (i.e.  $\ell \cap S \neq \emptyset$ ).
- (iv)  $S$  “avoids” the light-cones, i.e. no nonempty segment of a photon-geodesic lies inside  $S$ .  
(Note that any open segment of a geodesic is a geodesic again.)
- (v) there is no time-like curve connecting two points of  $S$ .
- (vi) there is no time-like geodesic connecting two points of  $S$ .

Note that (i)–(iii) are “closure conditions”, i.e. they try to make  $S$  big, while condition (iv) points in the direction that  $S$  is only  $n - 1$ -dimensional (in some sense), hence it tries to make  $S$  “thin” like a usual surface.

In the pictures we start out from the origin  $\bar{0}$  and try to build a simultaneity containing  $\bar{0}$  first by moving along the  $\bar{y}$ -axis and then by moving along the negative  $-\bar{x}$ -axis. Then we try to combine the two. While the figure does not prove the nonexistence theorem, it illustrates ideas about its plausibility. For more careful formulation and proof of non-existence of global time in GU cf. [15, p.263 (written by Malament), pp.269–287], Hawking-Ellis [18, p.170]. Earman [9, Lemma 4.1] is also (remotely) relevant here, but it proves less than what Gödel claims, namely, we do not require  $S$  to satisfy all properties of a Cauchy hypersurface (cf. [9, p.44] for definition of Cauchy hypersurfaces).<sup>11</sup>

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<sup>11</sup> The general relativistic computer constructed in Etesi-Németi [11] (cf. also Hogarth [20], Earman [9], Németi-Dávid [27]) can be realized in the Gödel-type universes, too, because of their special causal structure. This is interesting because we do not know whether the GU's enjoy the so called Malament-Hogarth property (cf. the literature on this definition, e.g. Earman [9], Malament-Hogarth [20]).

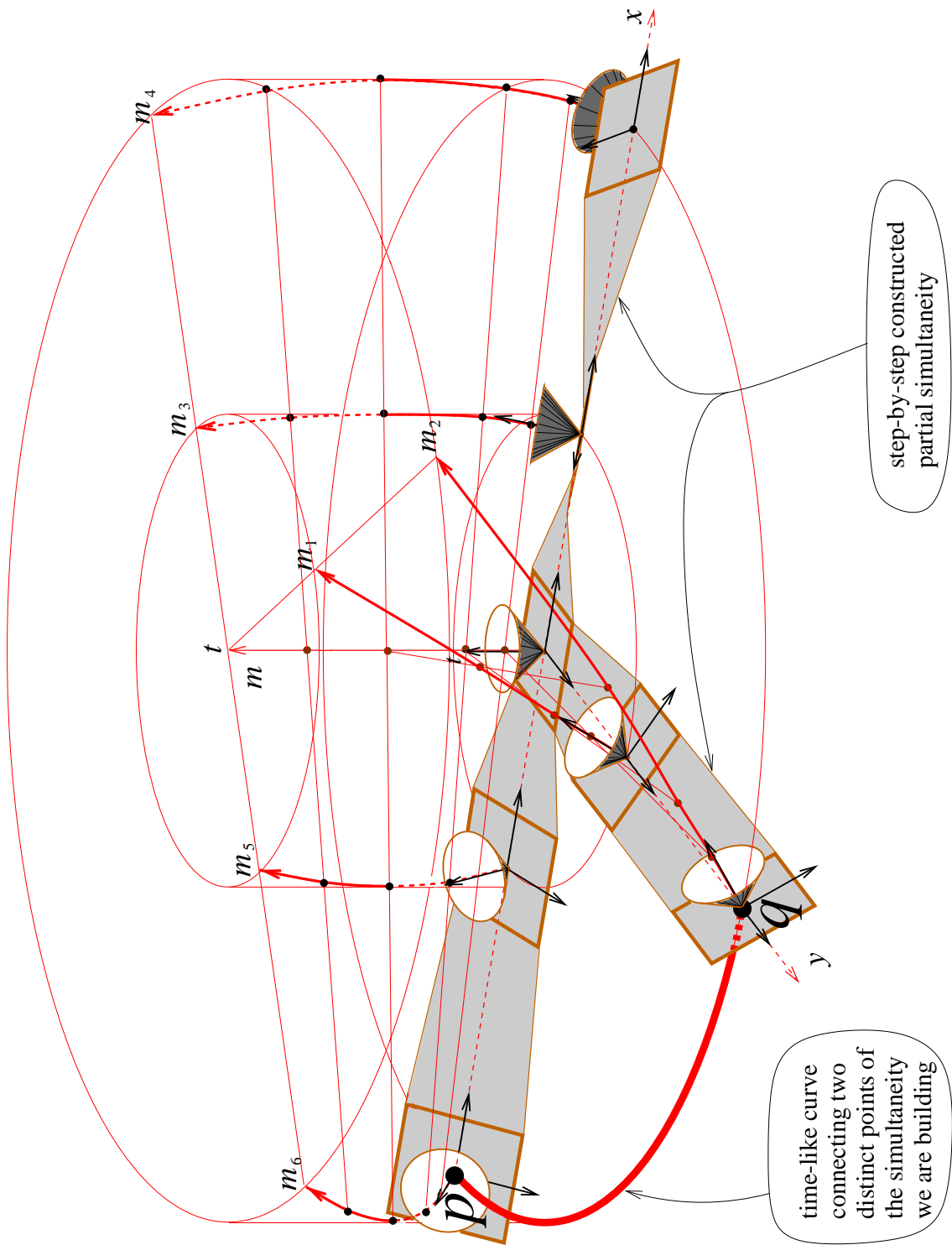


Figure 15: Idea of “non-foliation” of Gödel’s space-time. I.e. nonexistence of a global, natural simultaneity (or global time) in Gödel’s universe. See explanation on p.19.



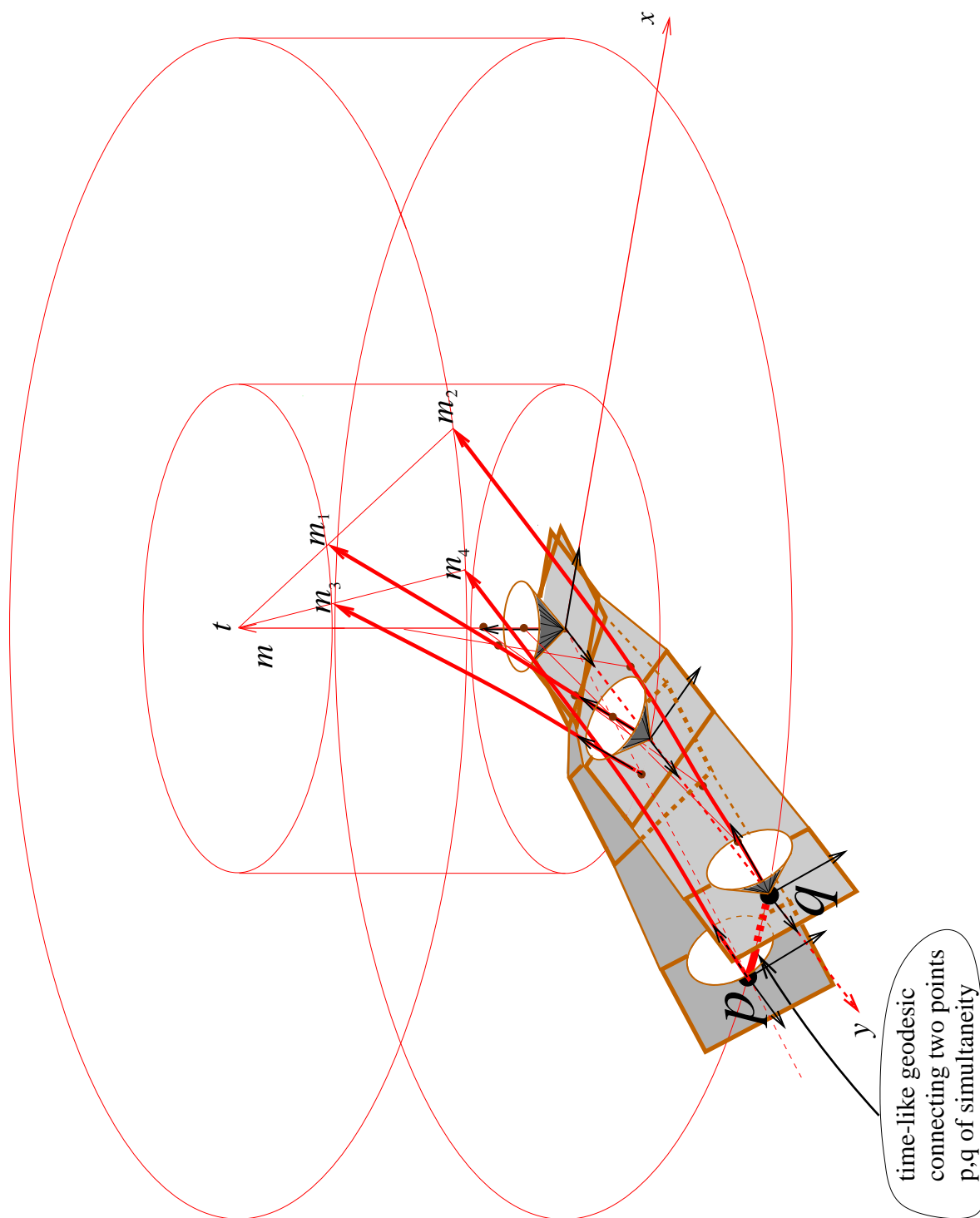


Figure 16: Previous figure but with the two strips of constructed simultaneity closer to each other,  $p, \bar{0}$  and  $q, \bar{0}$  are still simultaneous. The “informal logic” of these two figures generates a simultaneity connecting all points of space-time with each other. This is in contradiction with the intuitive notion of simultaneity.

## 4 Gödel's universe in co-rotating coordinates, “whirling dervishes”. Transforming the rotation away.

Gott [16, p.91] writes “You could equally well view Gödel’s universe as static and non-rotating, as long as self-confessed “nondizzy observers” would be spinning like whirling dervishes with respect to the universe as a whole.”<sup>12</sup> Below we will introduce new coordinates  $\langle T^r, X^r, Y^r, Z^r \rangle$  co-rotating with the matter content  $m_0, \dots, m_i, \dots$  of the universe. In  $\langle T^r, \dots \rangle$  the massive bodies  $m_i$  appear as static with their life-lines vertical lines. We will call  $\langle T^r, \dots \rangle$  “*Dervish World*” motivated by the above quotation from Gott. The transformation between the old spiral coordinates and the new rotating coordinates is elaborated later, on pp.70–73.

In the Spiral World, the “galaxies”  $m_1, m_2, \dots, m_i$  appear as rotating around  $m_0$  in direction  $\varphi$  with angular velocity  $\omega$  while their cosmic compasses  $x_i, y_i$  appear fixed (non rotating). As a contrast, the Dervish World shows  $m_1, \dots, m_i$  as motionless, while it shows their cosmic compasses as rotating in direction  $-\varphi$  with angular velocity  $\omega$ .

We will indicate on page 53 how this dervish world can be used to show that GU can be used to demonstrate that General Relativity (in its present form) does not imply the full version of Mach’s principle.

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<sup>12</sup>Gödel [13, p.271] writes: “Of course, it is also possible and even more suggestive to think of this world as whirling dervishes with respect to the universe as a whole.”

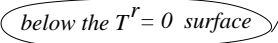


Figure 17: Gödel's universe GU in *rotating* coordinates  $T^r = t$ ,  $X^r$ ,  $Y^r$ . These coordinates co-rotate with GU, hence GU appears as *being at rest*. As a price, the local coordinate systems like  $\langle t', x', y' \rangle$  appear as rotating backwards (in direction  $-\varphi$ ) in the new coordinate system. The transformation between the old spiral coordinates and new rotating ones is elaborated on p.70.

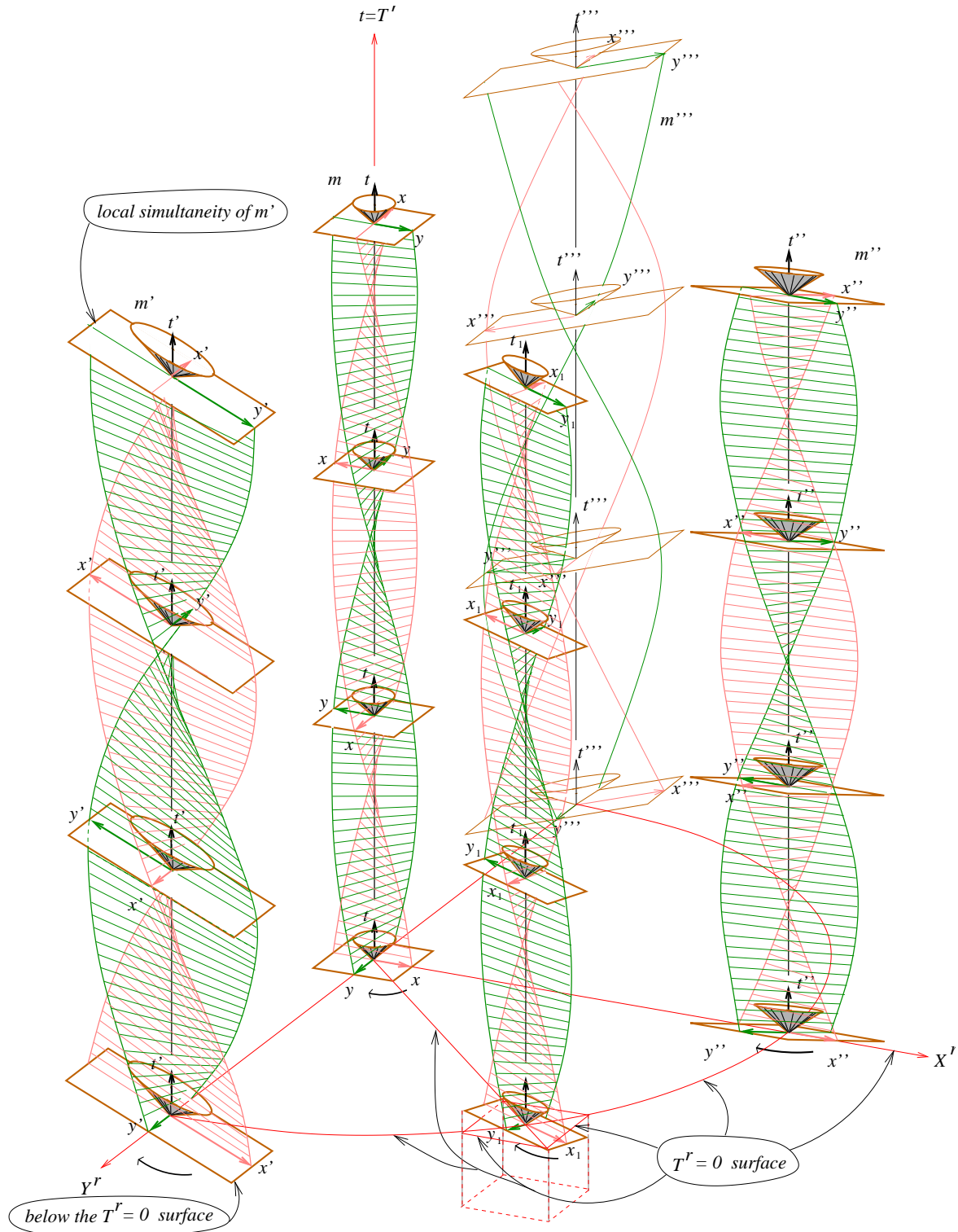


Figure 18: We have a system of static, non-moving massive observers  $m, m', m''$  etc. (the same as in Figures 7–9) whose cosmic compasses i.e. whose local coordinate systems are spinning around creating a whirling effect. Gott [16, p.91] called these “whirling dervishes”. This arrangement can be used to show that Mach’s principle is violated. See p.53 for explanation.

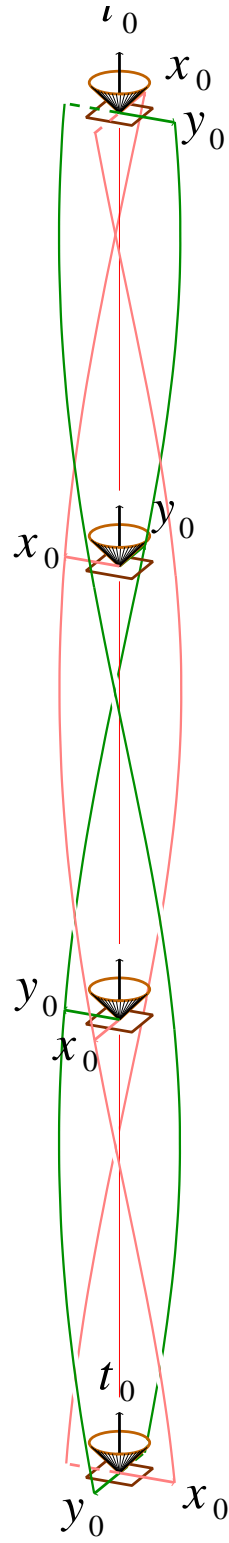


Figure 19: A typical dervish consisting of massive observer (or galaxy)  $m_0$  and its cosmic compasses  $\langle x_0, y_0, z_0 \rangle$ . In other words,  $m_0$ 's dervish is  $m_0$ 's *local* coordinate system.  $\omega = \pi/15$ .

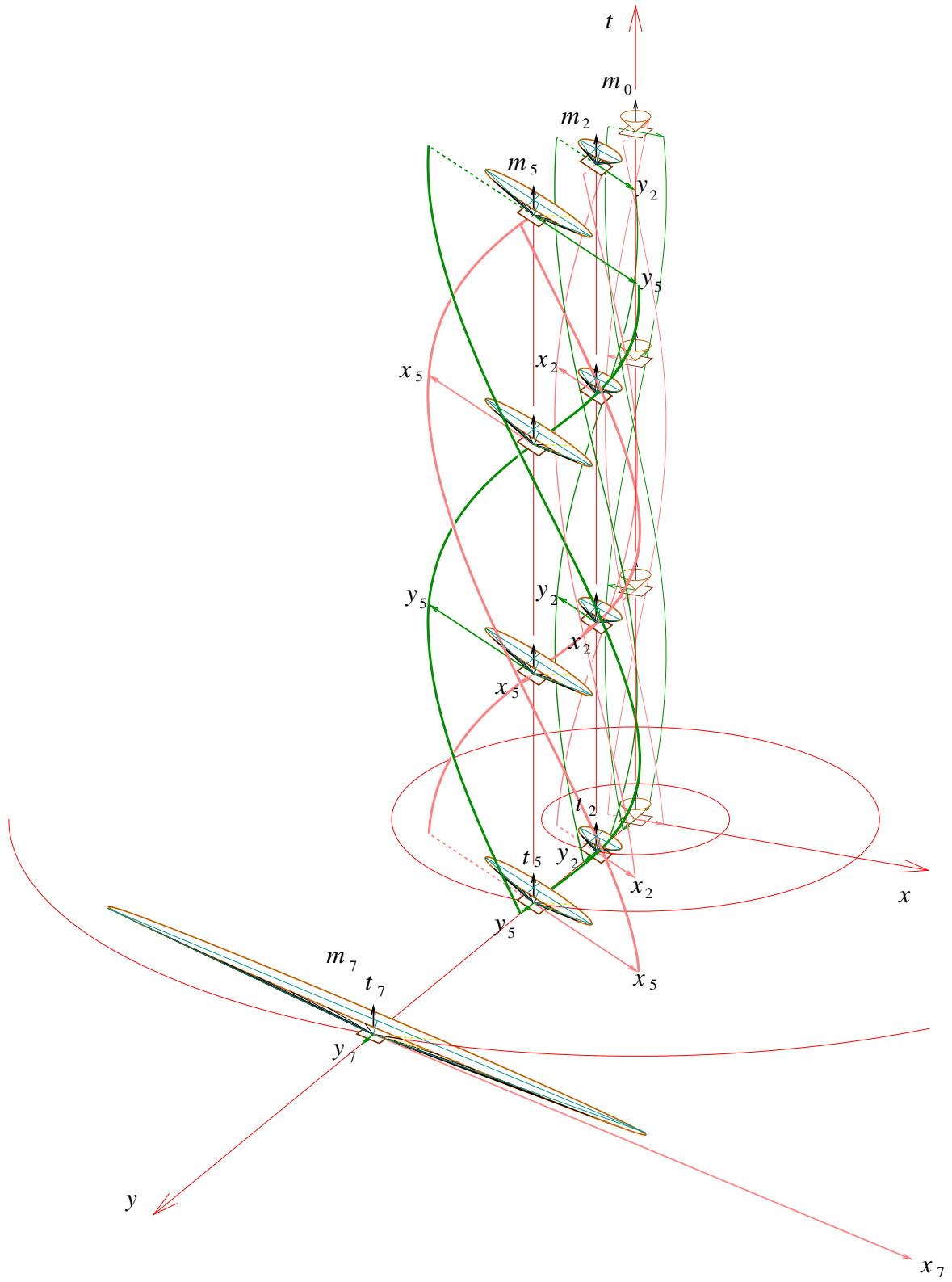


Figure 20: Dervishes  $m_0, \dots, m_7$  involving greater radiuses, hence more “violent” whirling effects.  $\omega = \pi/15$ . Re-calibrated version of Map 2 applies, cf. p.81.

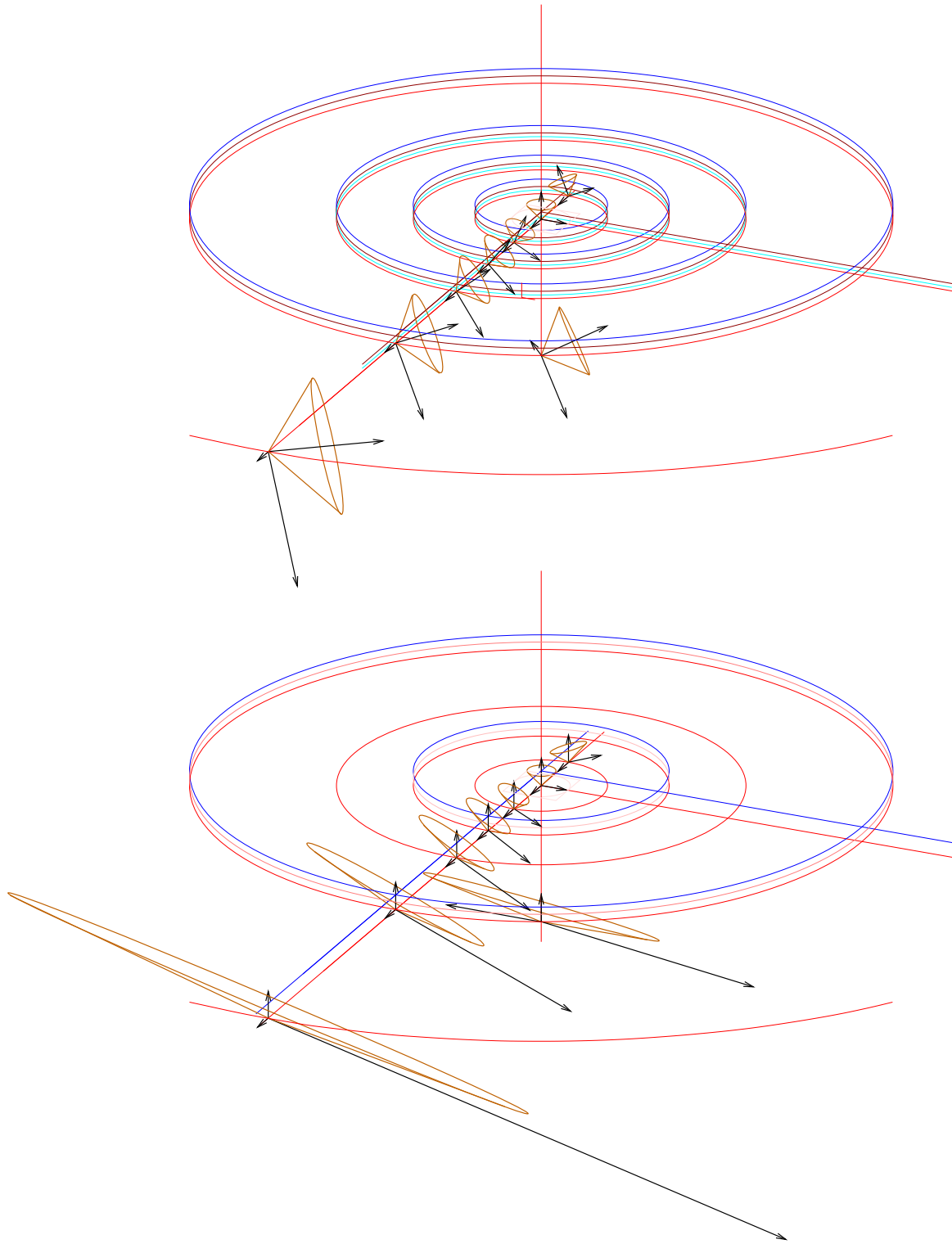


Figure 21: Light-cones and local unit vectors of spiral world above, and their counterparts in dervish world  $\langle T^r, \dots, Z^r \rangle$  below. Detailed representation of upper part is in Figures 12, 13, 14 and that of lower part is in next Figure 22. See also Figures 17-20. The transformation between the two worlds is described on pp.70-73.

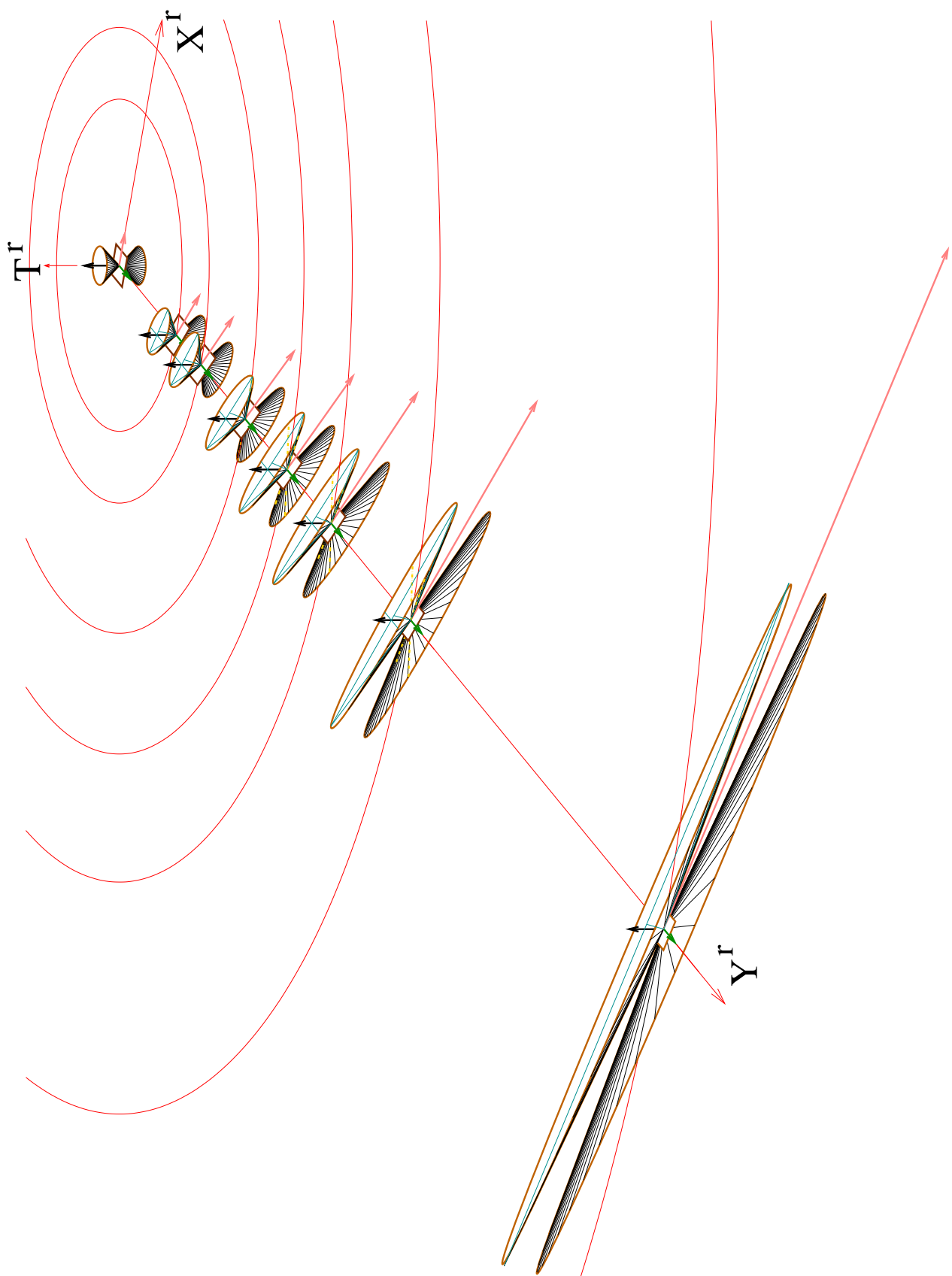


Figure 22: Light-cones with local unit vectors in dervish world  $\langle T^r, \dots \rangle$ .  $\omega = \pi/30$ , Map 2 applies.



## 5 Fine-tuning the space-time structure of the Naive GU obtained so far. Tilting the light-cones.

First we show two pictures hinting at the fact that the lengths of unit-vectors etc. in our Naive Dervish World might be of inconvenient proportions.

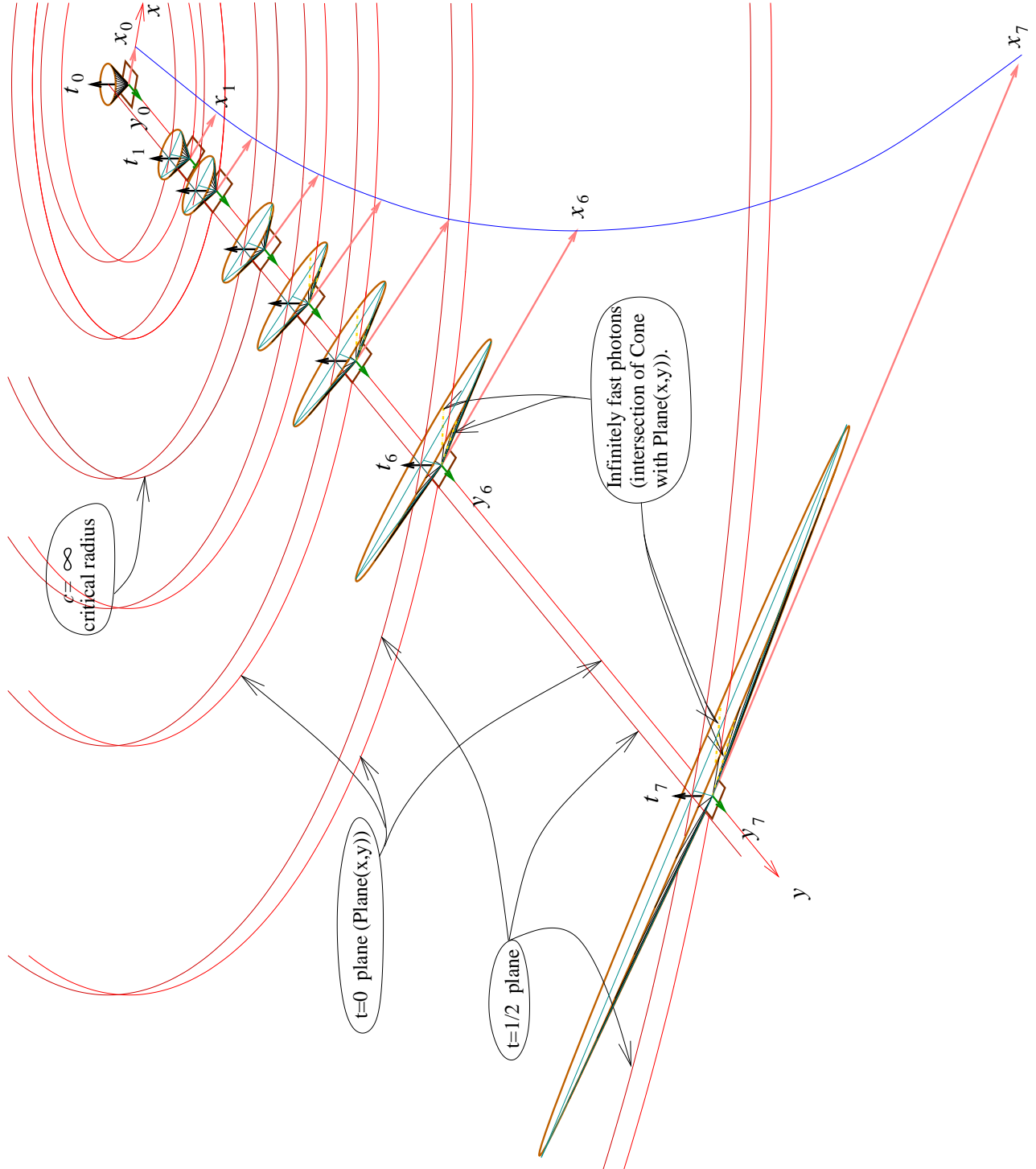


Figure 23:  $\omega = \pi/30$ , Map 2 applies.

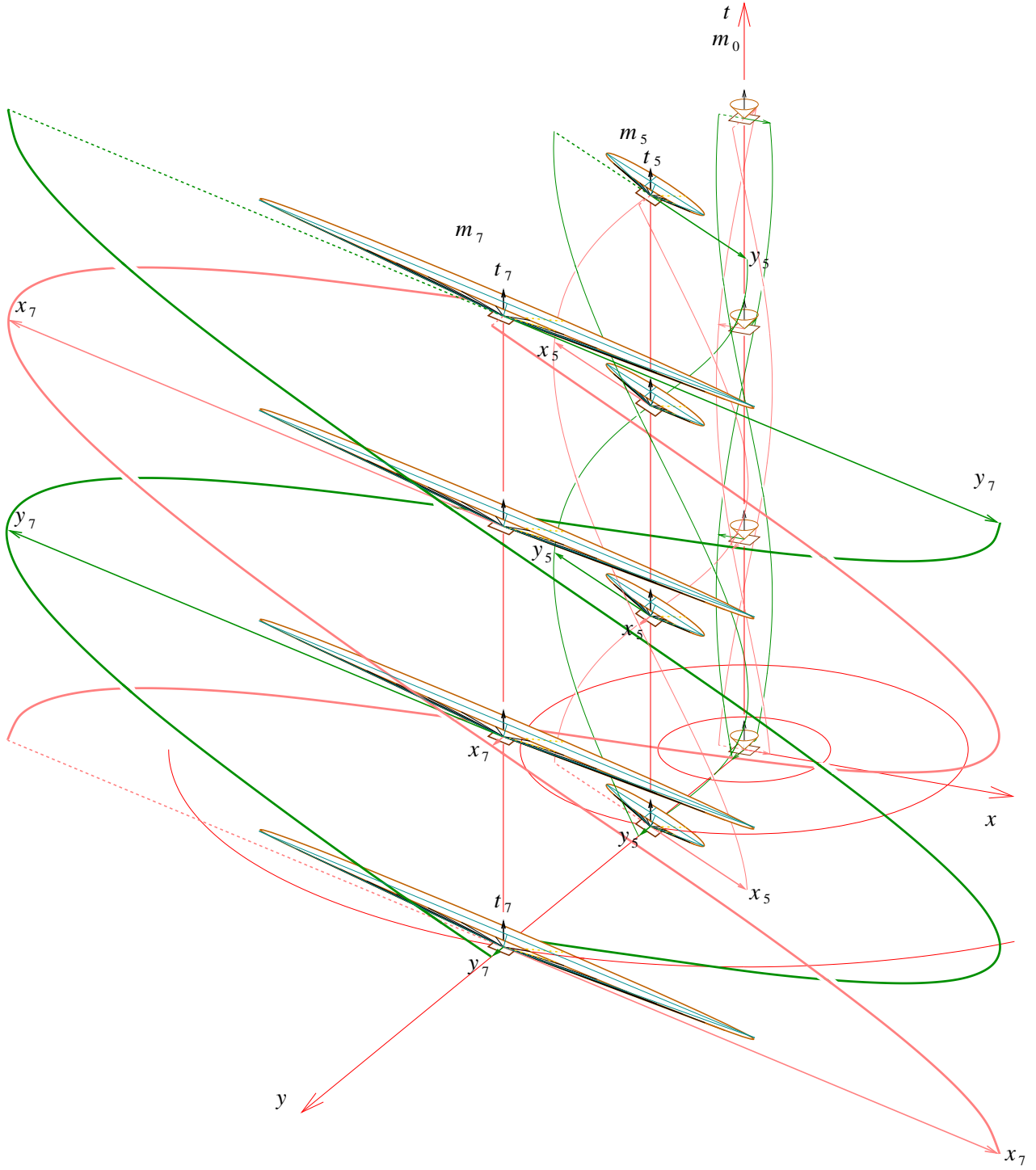


Figure 24: Whirling dervishes on larger radii. Re-calibrated version of Map 2 applies as follows.  $r'(m_i) = 2 \cdot r(m_i)$ ,  $v'(m_i) = v(m_i)$ ,  $\omega' = \omega/2$ ; where  $r', v', \omega'$  belong to the present figure while  $r, v, \omega$  belong to Map 2.

The fact that the  $x_i$  vector of  $m_i$  has a much longer component parallel with coordinate  $X^r$  than  $x_0$  (illustrated in the previous two figures) is the visual manifestation of the following fact, seen better in the spiral world. In the spiral world,  $m_i$  can send a photon  $ph$  upward almost parallel with the  $t$  axis such that  $ph$  reaches  $m_i$  again in a “rigidly bounded” time (an upper bound is  $4\pi/\omega$ ) where the bound is independent of the choice of  $i$ . We choose the path of  $ph$  such that its distance from  $m_0$  remains constant (ly the  $m_0$ – $m_i$  distance). This path need not be geodesic but as Gödel wrote, we can use mirrors to force  $ph$  to follow this path. See Figure 25.

In Gödel’s Universe the return-time of the photons sent around  $m_0$  in a circle of radius  $r$  tend to infinity as  $r$  tends to infinity.

Let us see how we can remove this difference with Gödel’s universe without destroying the logic of our construction. How can we fine-tune our construction? We are aiming at the “smallest” and simplest change so that the logic of our construction would remain intact. Changing the length’s of the  $x_i$  vectors and keeping the other unit-vectors as they were results in making the light-cones narrower. Since this will not lead to CTC’s, we will “tilt” the light-cones, instead. So, in fine-tuning the Naive GU we will speak about tilting the light-cones, and we will call the new space-time Tilted GU.

Let us work in the dervish world.

Choice 1 We can tilt the light-cones forwards (in the positive  $\varphi$  direction) such that with increasing  $r$  (radius) we also increase the tilting. This can be done in such a manner that the difference we talked about disappears. The result of such tilting results a version of NGU represented in Sections 5-6 (Figures 28–45). The so obtained tilted universe resembles very closely the universes presented in Gödel’s papers. (E.g. they agree in many structural properties [in Gödel’s sense].)

Choice 2 We can also tilt the light-cones (in dervish world) backwards, opposite to the  $\varphi$  direction, carefully enough such that the difference goes away and we do not induce other undesirable effects. See Figure 27. This Choice 2 tilting is just Choice 1 tilting seen from another coordinate system (namely by using the coordinate transformation  $\varphi \rightarrow -\varphi$ ). Below we will explore Choice 1, and in Section 7 (p.52) we explore Choice 2. We will see that both Choice 1 and Choice 2 have their advantages.

From now on, we concentrate on Choice 1.

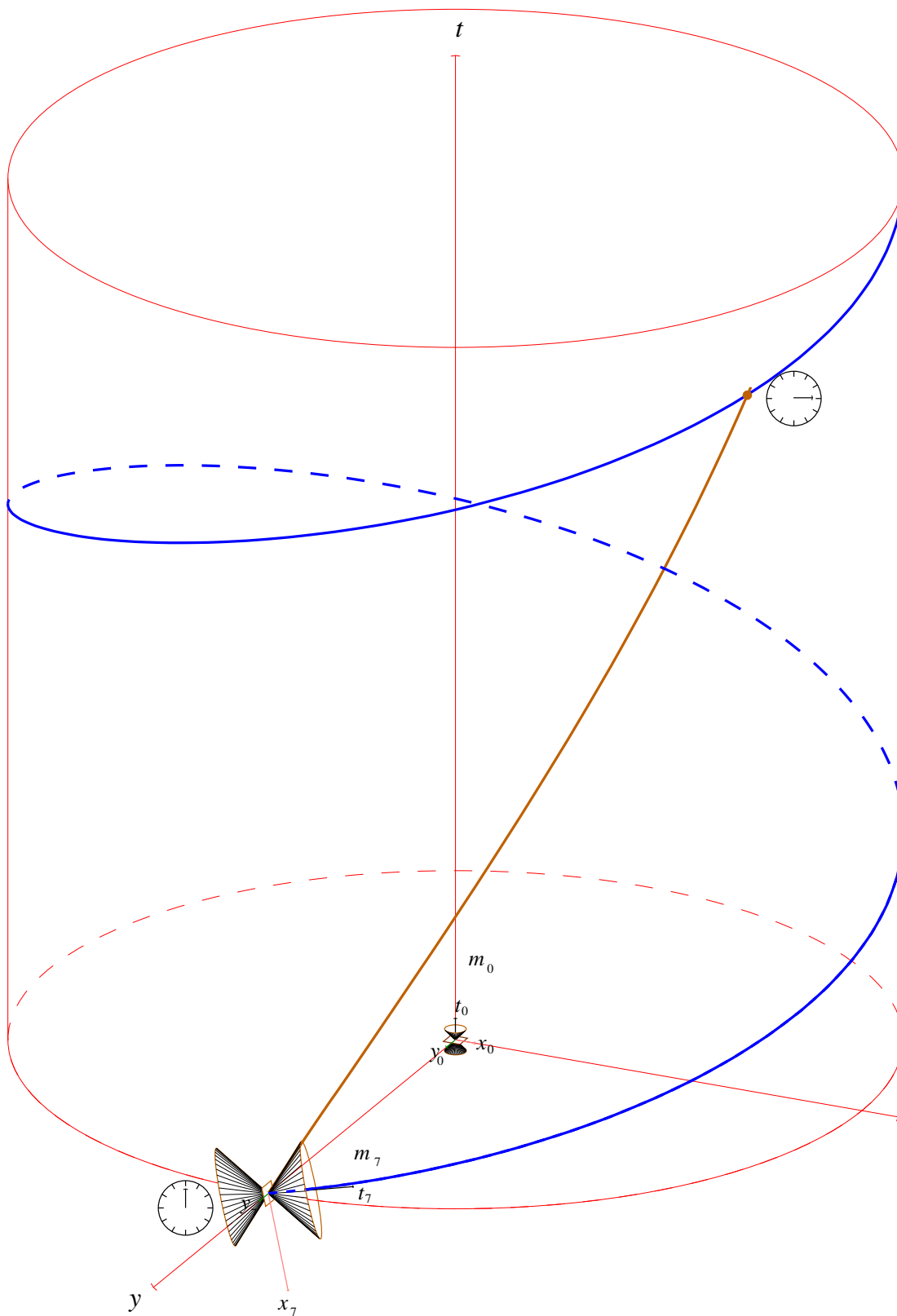


Figure 25: The time needed for a photon sent out by  $m_7$  and kept with mirrors on a circle around  $m_0$  to come back is a little more than the time needed for the universe to make a turn.

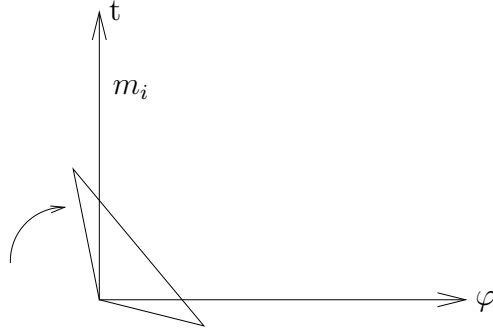


Figure 26: Choice 1 is that we tilt the light-cones forwards.

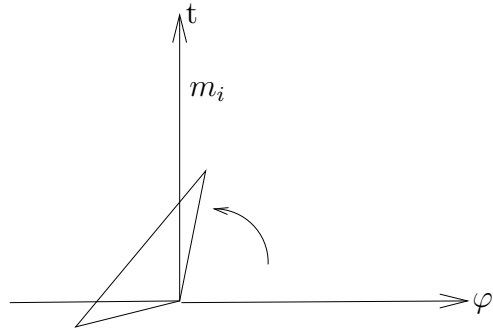


Figure 27: Choice 2 is that we tilt the light-cones backwards.

We will call the tilting in Choice 1 “*forward-tilting*”, the so obtained dervishes *tilted dervishes*, and the so obtained (tilted) dervish world *Tilted Dervish World* or *Choice 1 Dervish World*. Recall that we describe a simple transformation between the spiral world  $\langle t^s, \dots \rangle$  and the dervish world  $\langle t^d, \dots \rangle$  in Section 9 (p.70). We use this transformation for transforming the new, tilted universe from the dervish world to the spiral world. We call the result *Tilted Spiral World* or use simply the adjective “new spiral” or “refined-spiral” for referring to the so obtained light-cones as new spiral cones or back rotated ones. The expression “rotating back” or “back-rotating” intends to refer to application of the inverse transformation  $\langle t^d, \dots \rangle \longrightarrow \langle t^s, \dots \rangle$  described in Section 9. In such contexts the inverse transformation is applied to the result of forward-tilting.

The result of the above outlined forward-tilting is the Gödel-type universe which we will describe in more detail in the coming parts. We will call this space-time Tilted GU (or sometimes new GU). Instead of defining the tilting of the cones at each point, we will give details of the tilting for the cones occurring in the figures only. These tilted light-cones (with local unit-vectors) and their new spiral versions are depicted and constructed in detail in Section 9. These objects (light-cones,  $m_i$ ’s etc) are systematically arranged in space-time (i.e. are coordinatized) in Maps 1,2 (pp.80, 81). These maps also include angular velocities, tangential velocities.

In this section we describe “Tilted Dervish World”, and in the next section, Section 6, we describe “Tilted Spiral World”.

# Tilted dervishes (fuller description of new GU in dervish world).

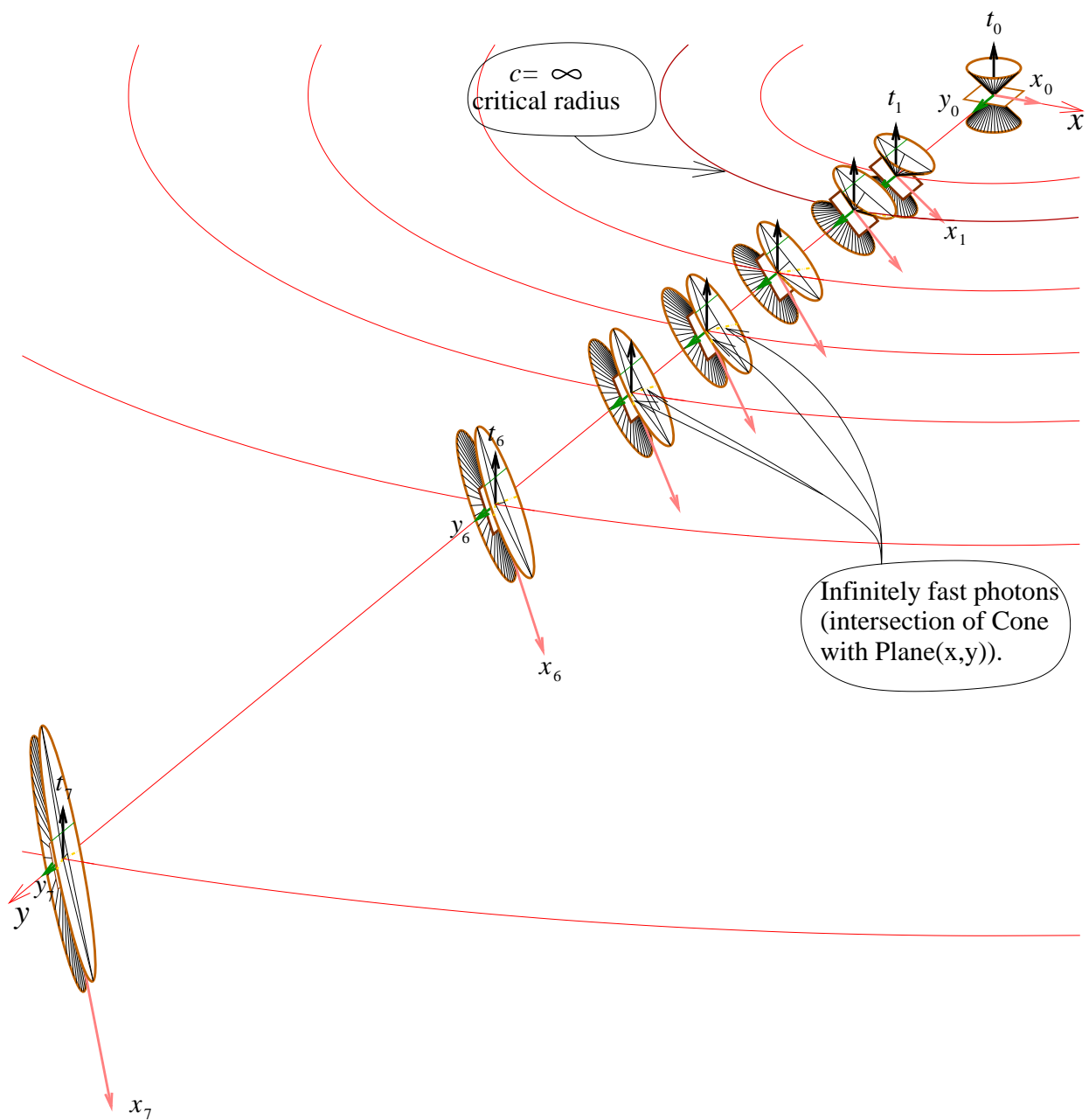


Figure 28: Tilted-dervish universe or Choice 1 Dervish World. Light-cones, local unit-vectors along the  $y$ -axis.  $\omega = \pi/30$ , Map 2 applies.

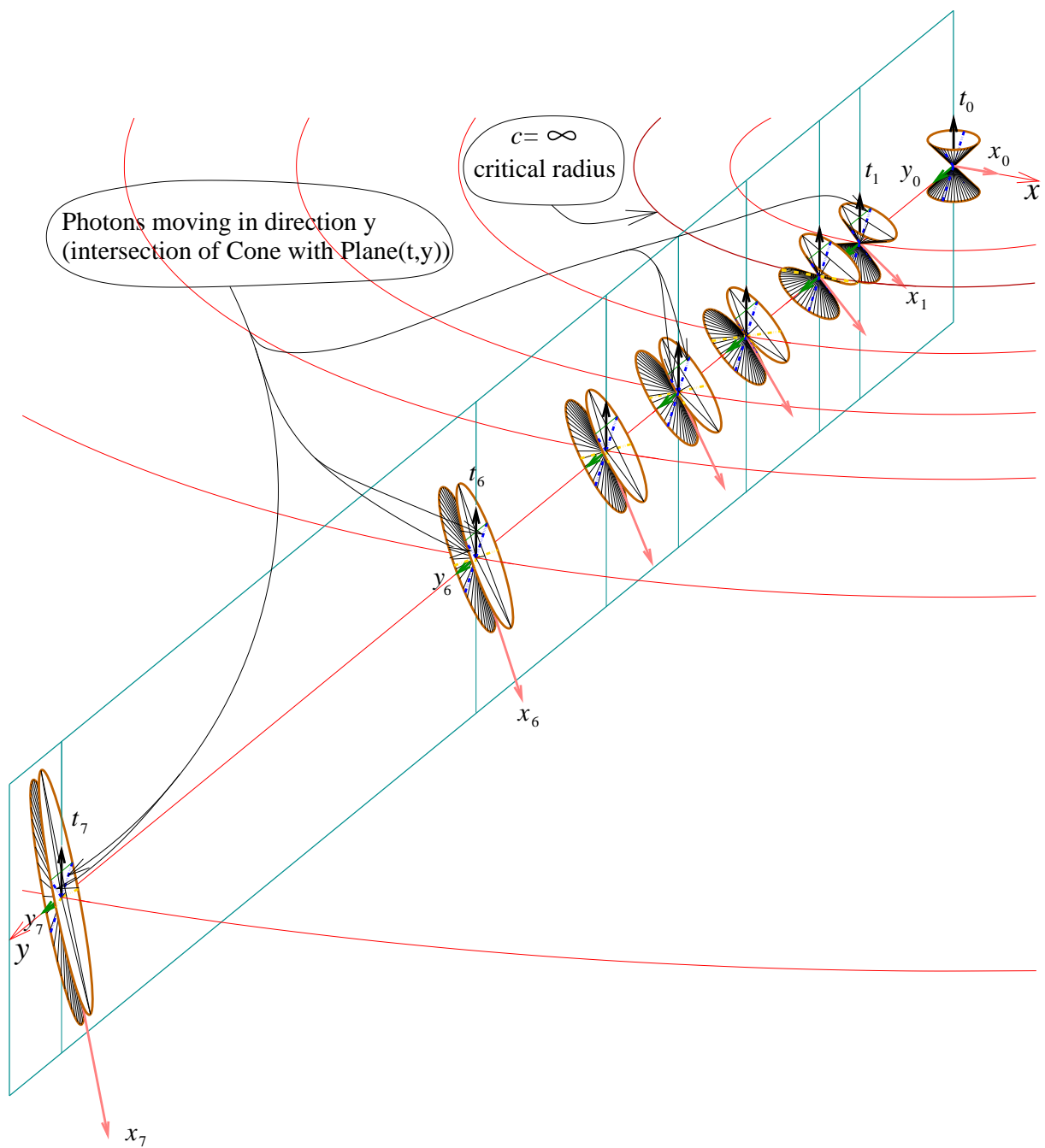


Figure 29: Tilted Dervish World (Choice 1 Dervish World).  $\omega = \pi/30$ , Map 2 applies.

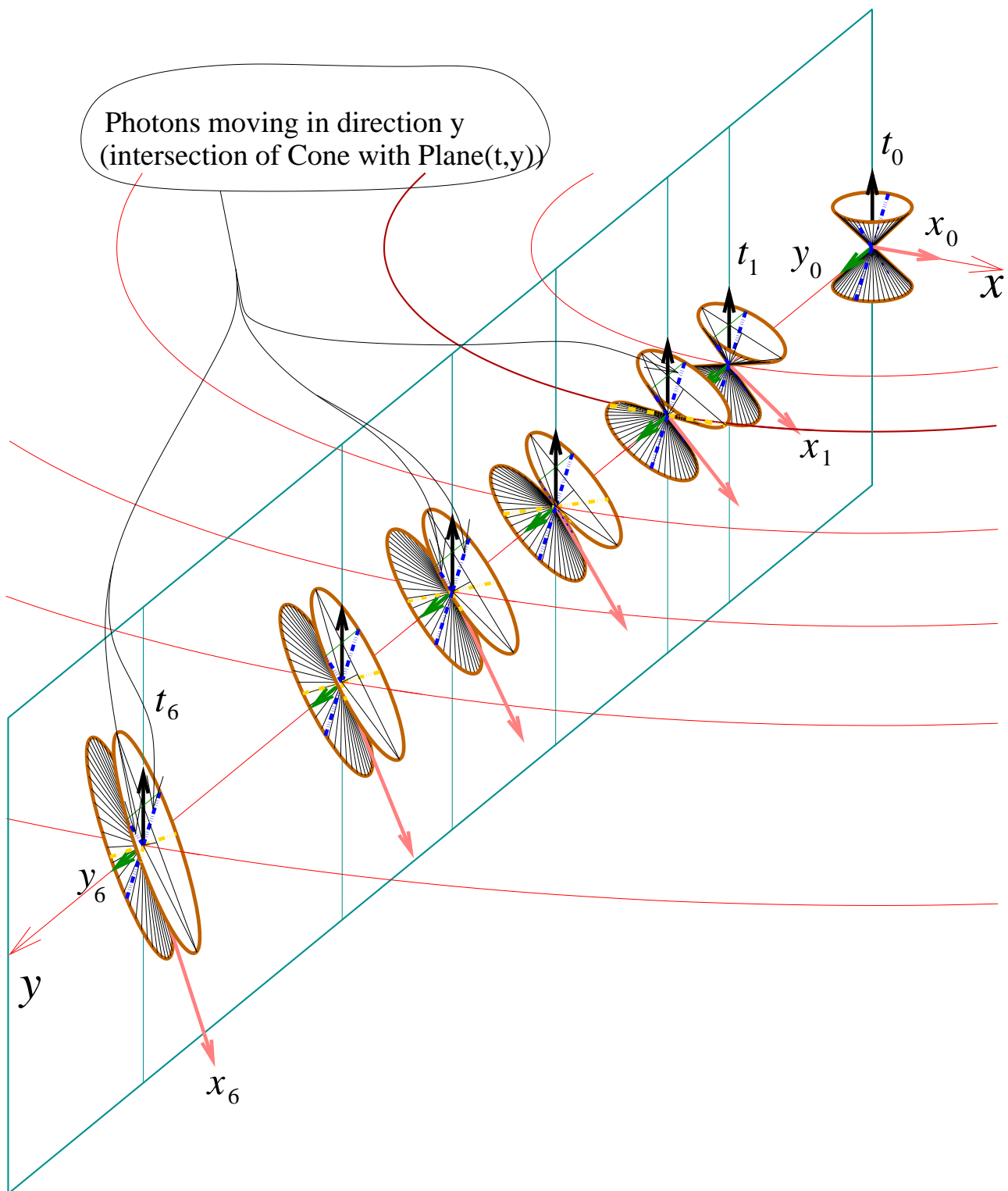


Figure 30: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.



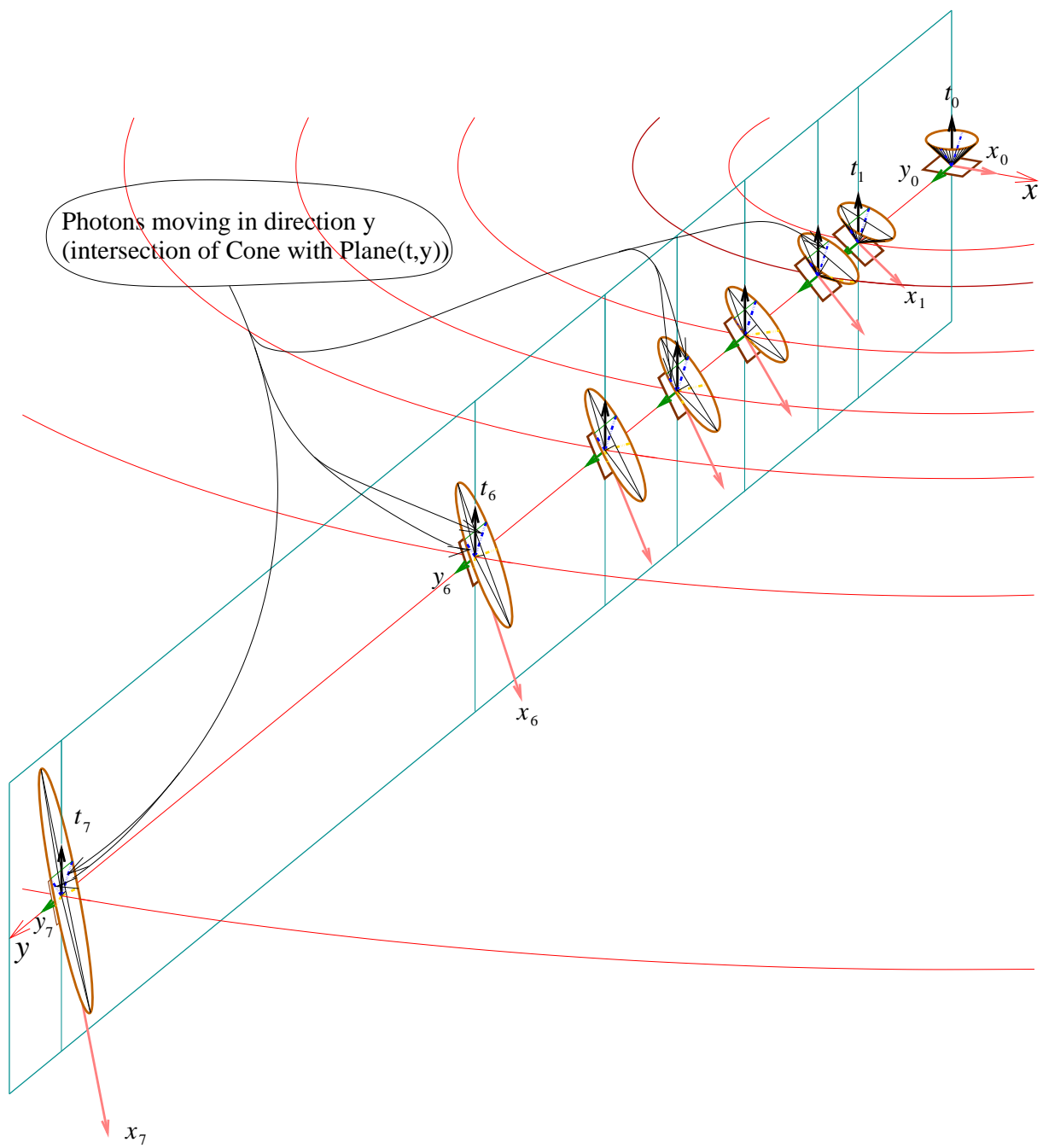


Figure 31: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.

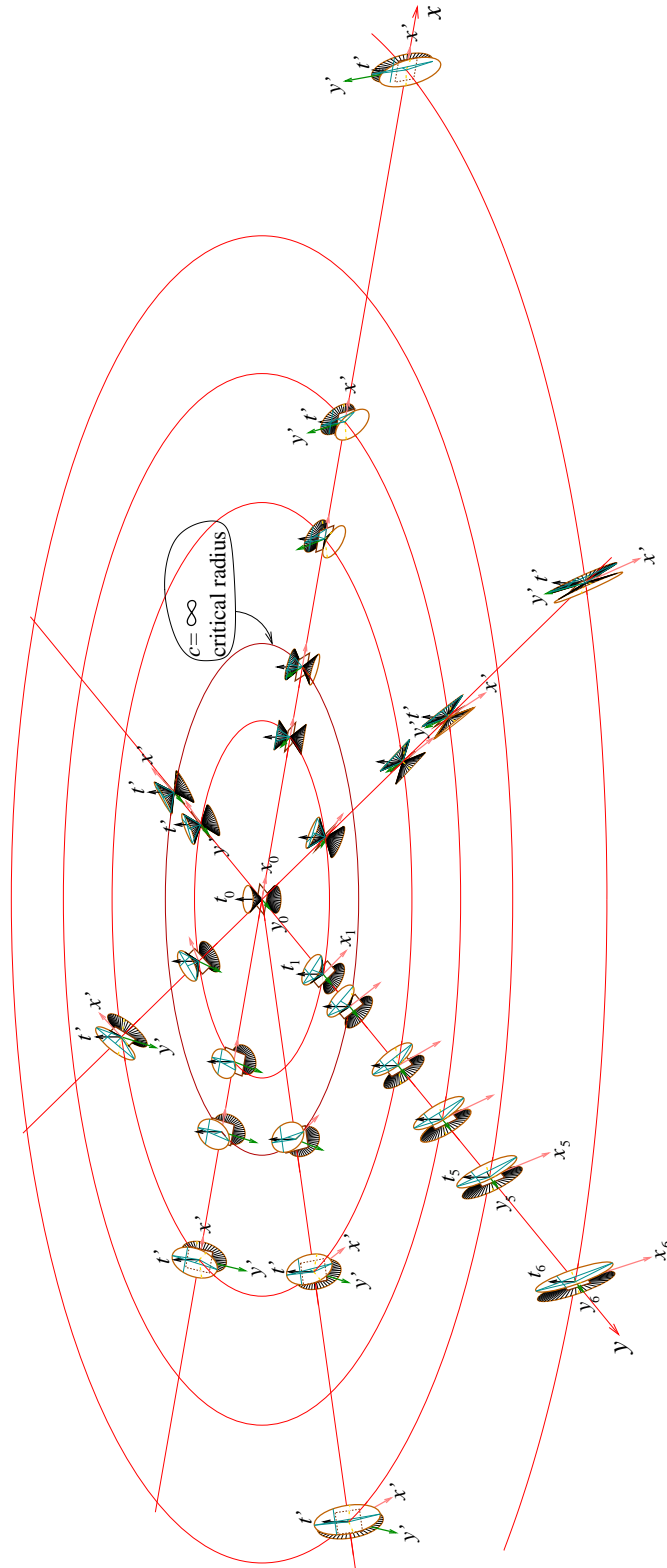


Figure 32: Tilted-dervish universe. (Choice 1 Dervish World.) Light-cones, local unit-vectors on the  $xy$ -plane.  $\omega = \pi/45$ , Map 1 applies.

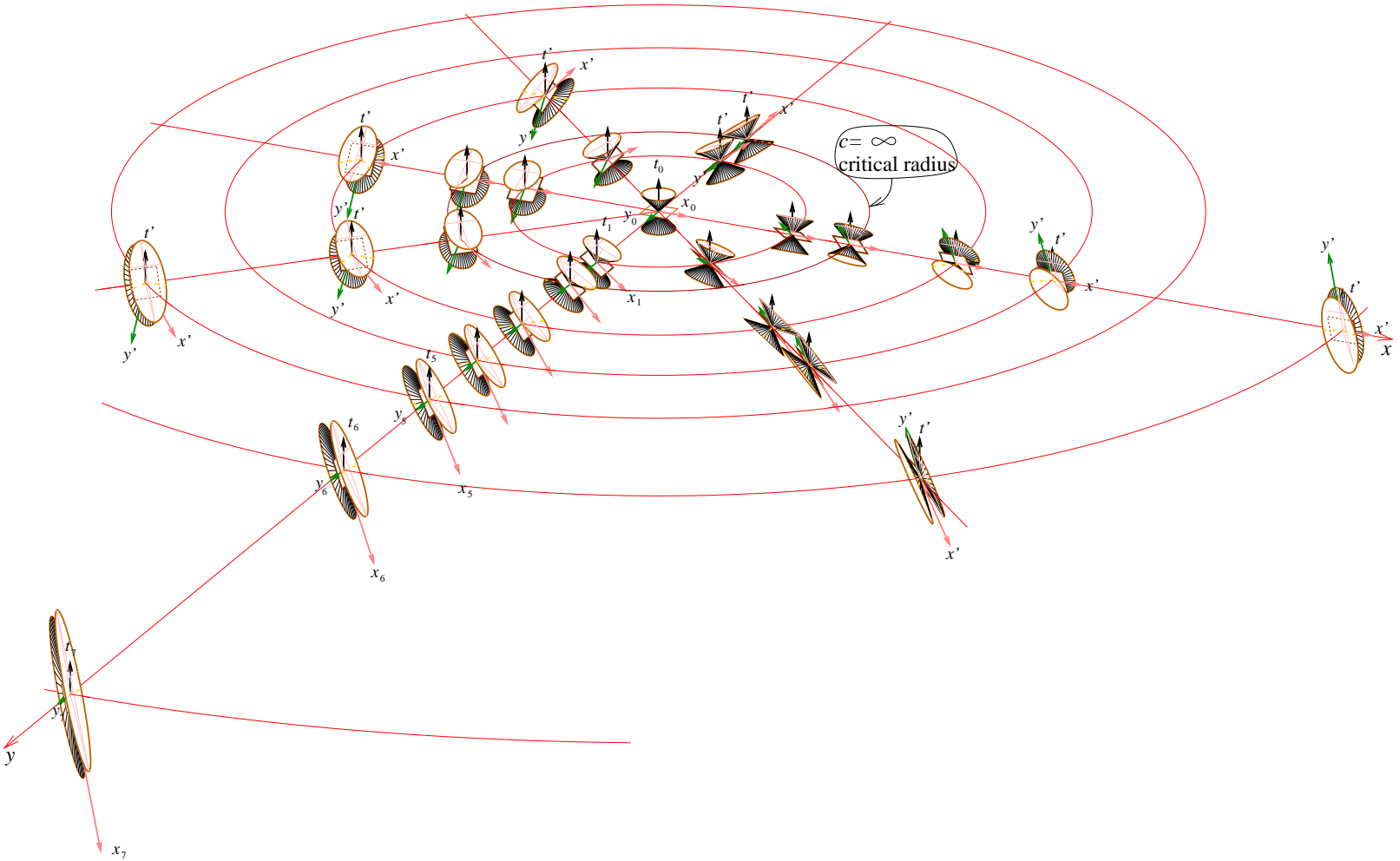


Figure 33: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.

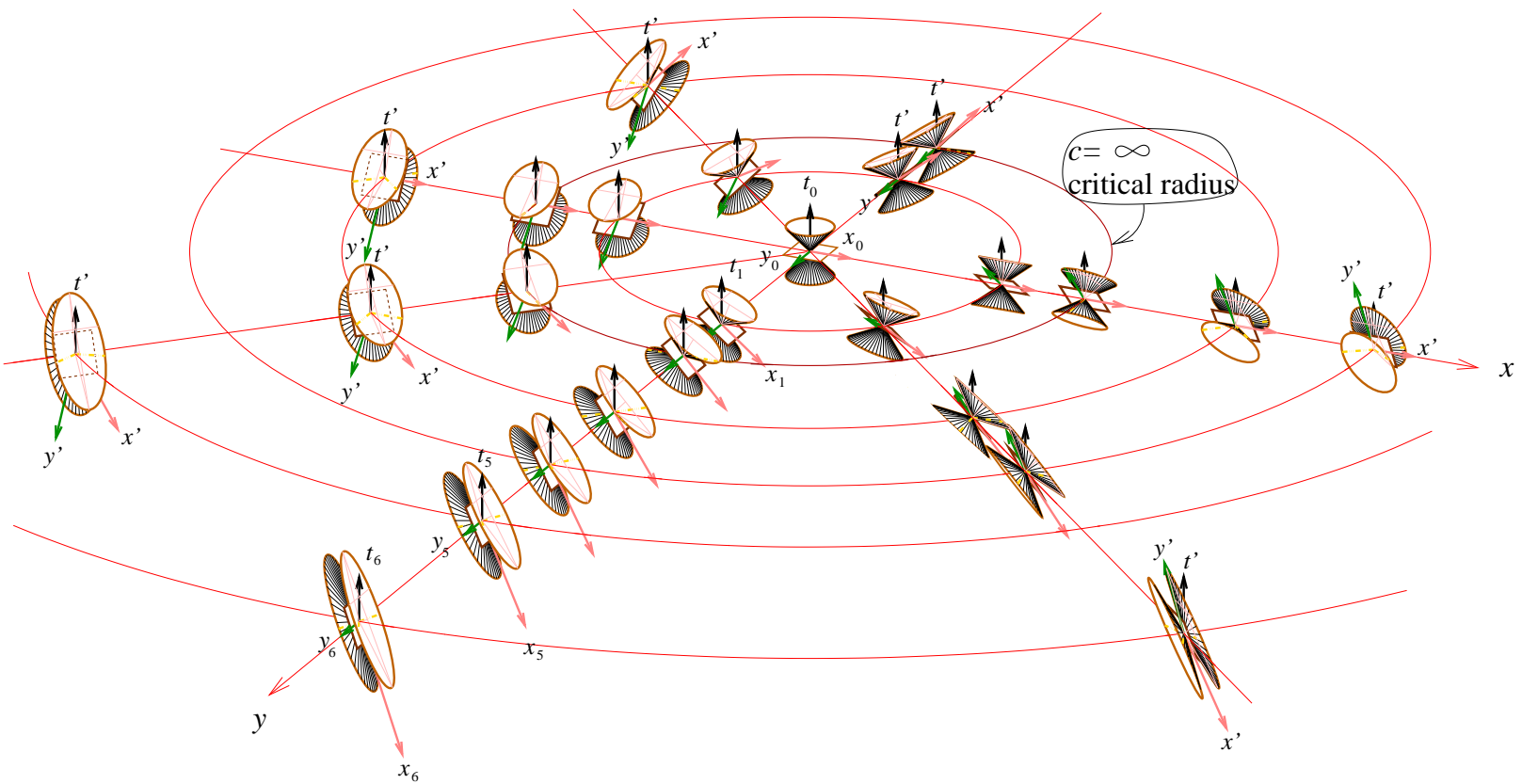


Figure 34: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.

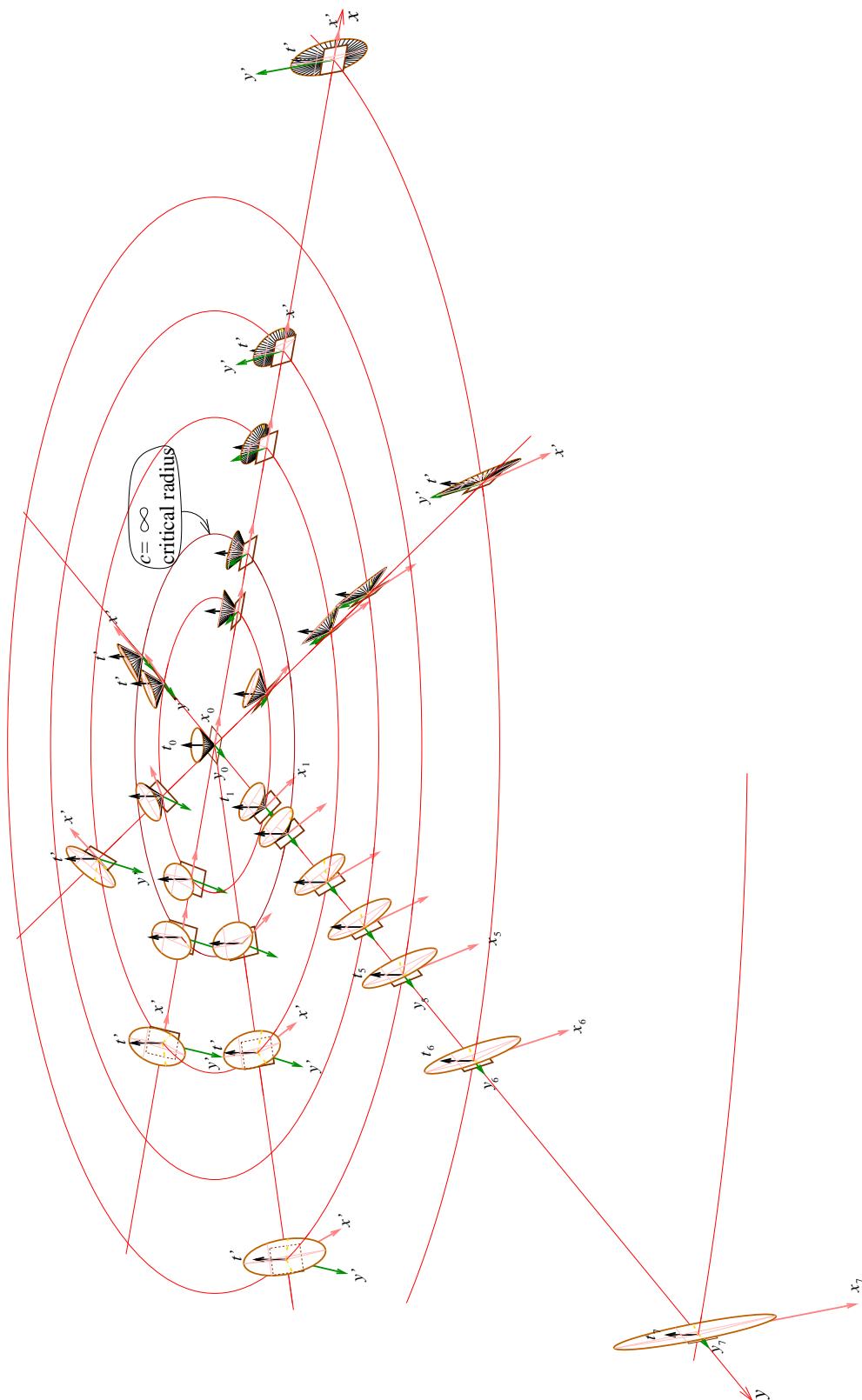


Figure 35: Tilted Dervish World. Compare with Figure 61 on p.169 in Hawking-Ellis [18] (cf. also Fig.1 herein).  $\omega = \pi/30$ , Map 2 applies.

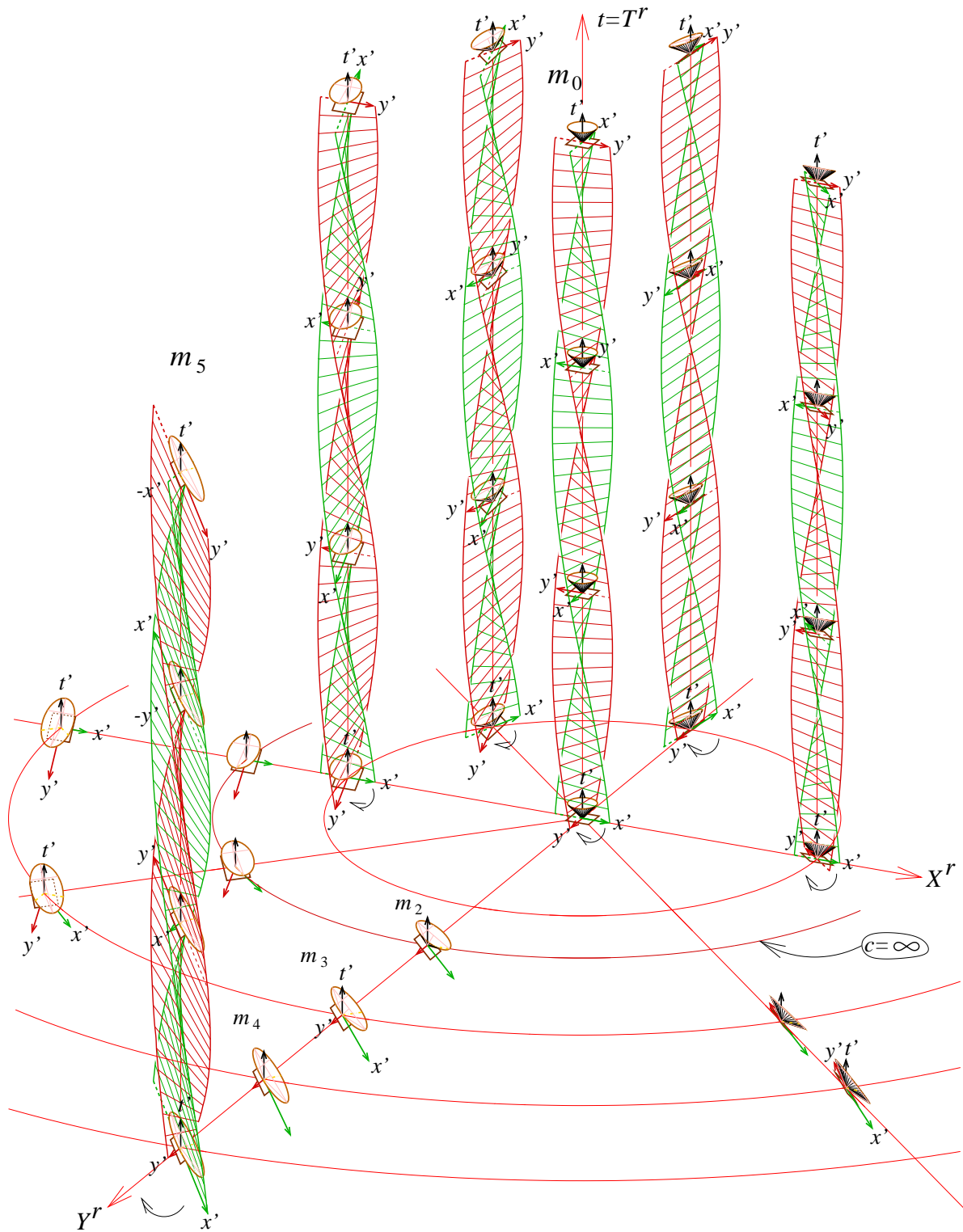


Figure 36: Tilted Dervish World. “ $\omega$  of universe” =  $\pi/60$  (recalibrated version of Map 2 applies). Spinning dervishes are artificially sped up (“artificial  $\omega$  of dervishes” =  $\pi/15$ ).

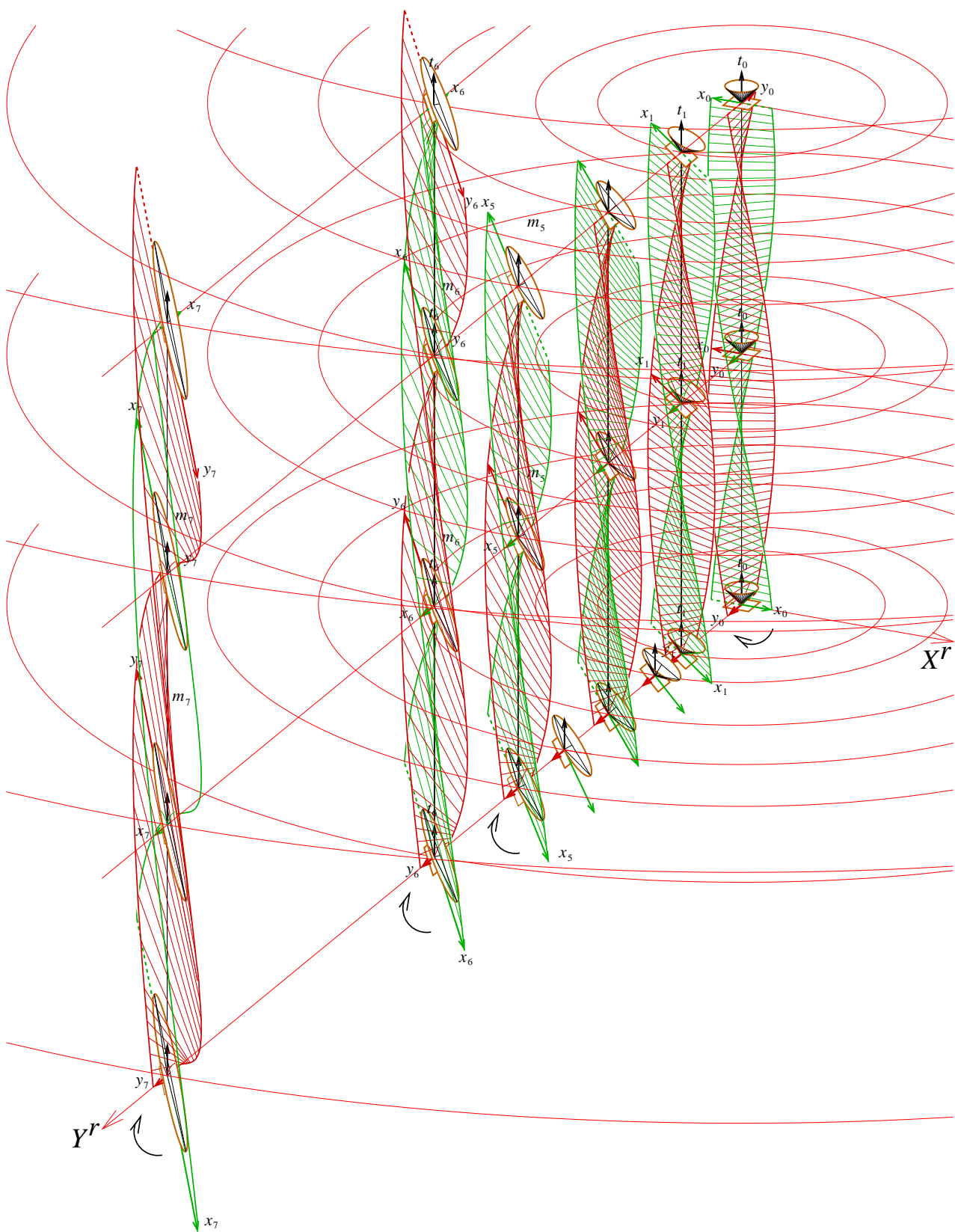


Figure 37: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.

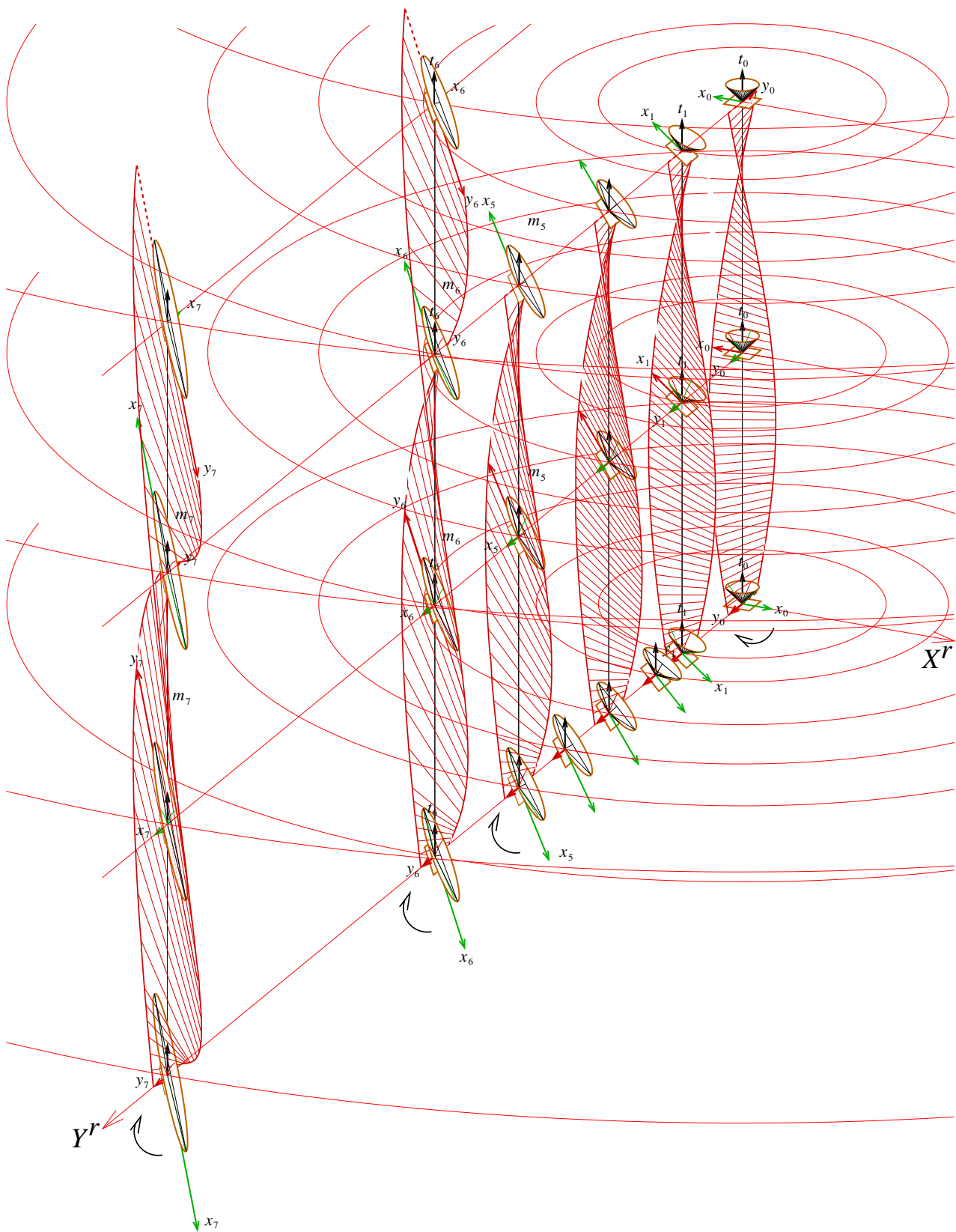


Figure 38: Tilted Dervish World.  $\omega = \pi/30$ , Map 2 applies.



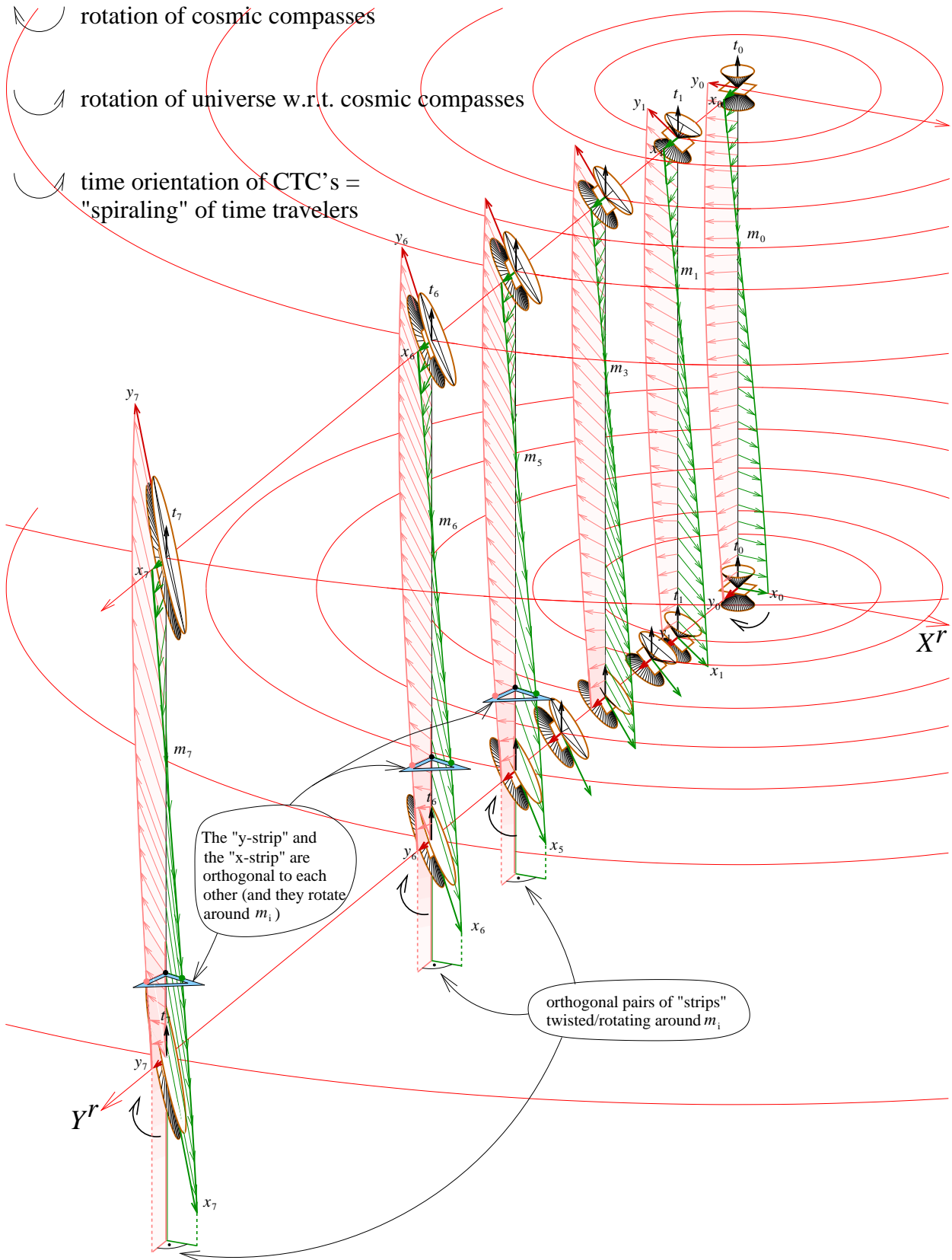


Figure 39: Tilted dervishes with original angular velocity.  $\omega = \pi/30$ , Map 2 applies.

## 6 Tilted Spiral World, i.e. Choice 1 Spiral World.

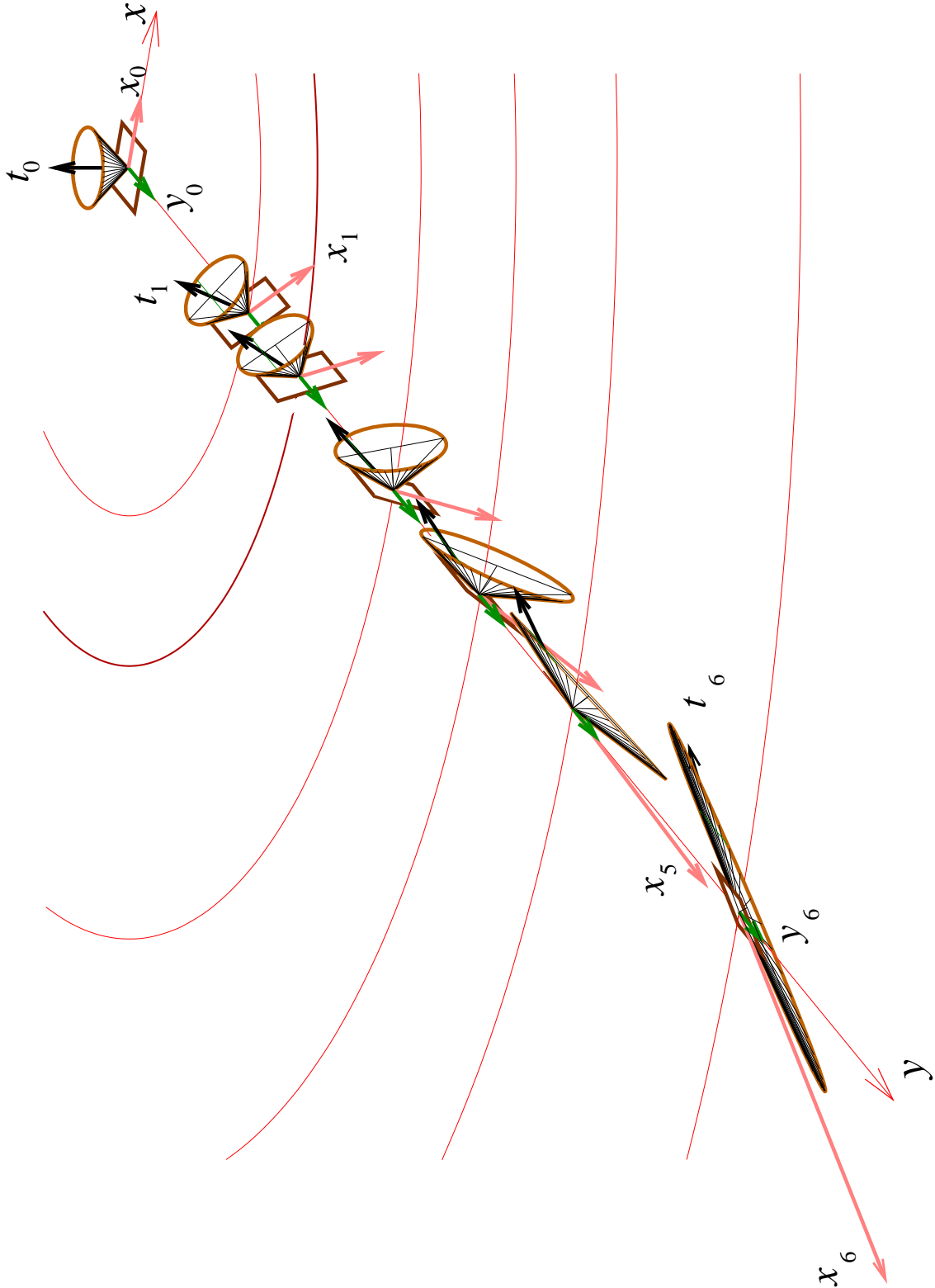


Figure 40: Tilted spiral world, i.e. Choice 1 Spiral World. Light-cones, unit-vectors along the  $y$ -axis.  $\omega = \pi/30$ , Map 2 applies.

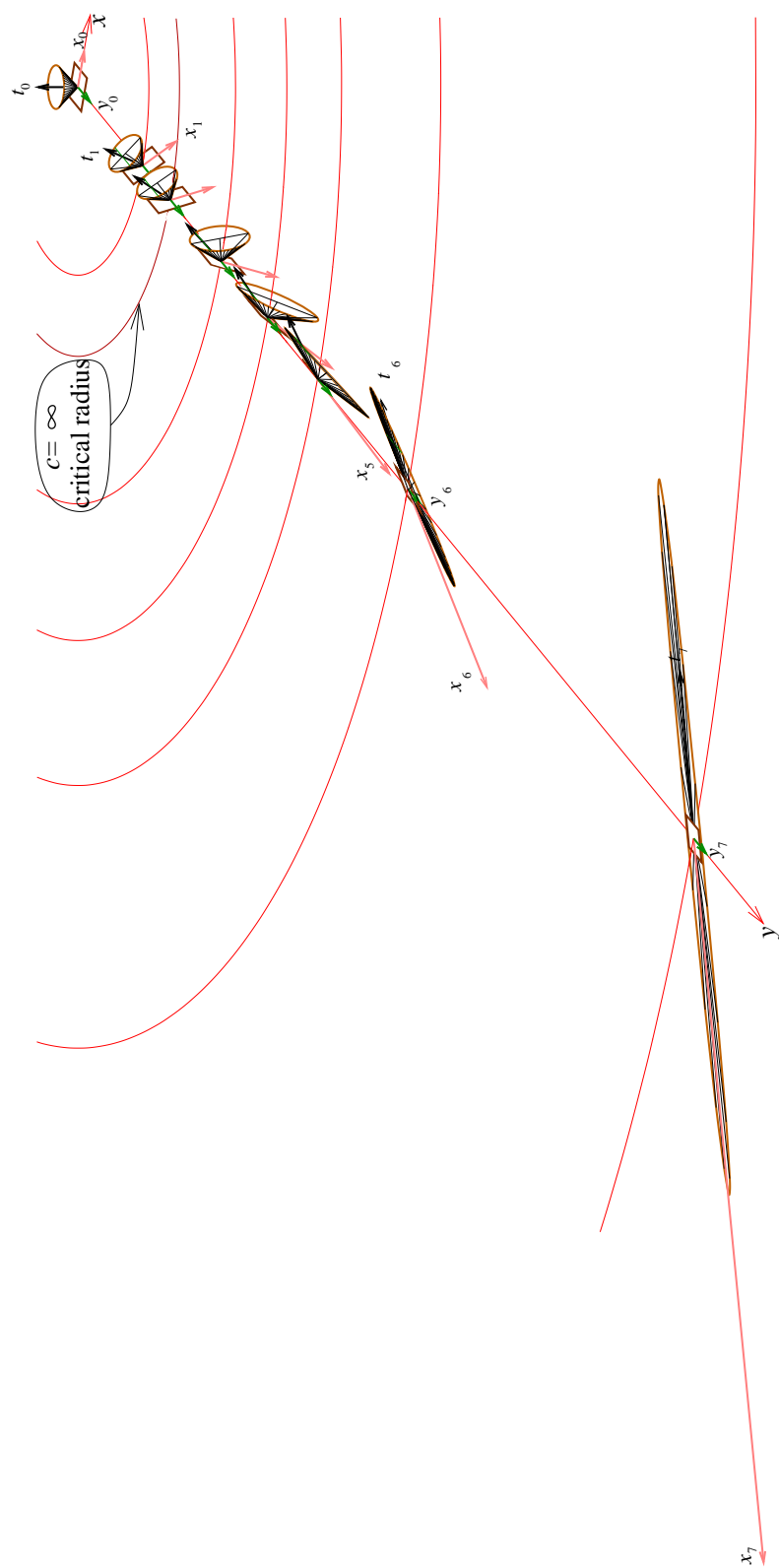


Figure 41: Tilted Spiral World.  $\omega = \pi/30$ , Map 2 applies.

Figure 42: Tilted Spiral World.  $\omega = \pi/30$ , Map 2 applies.

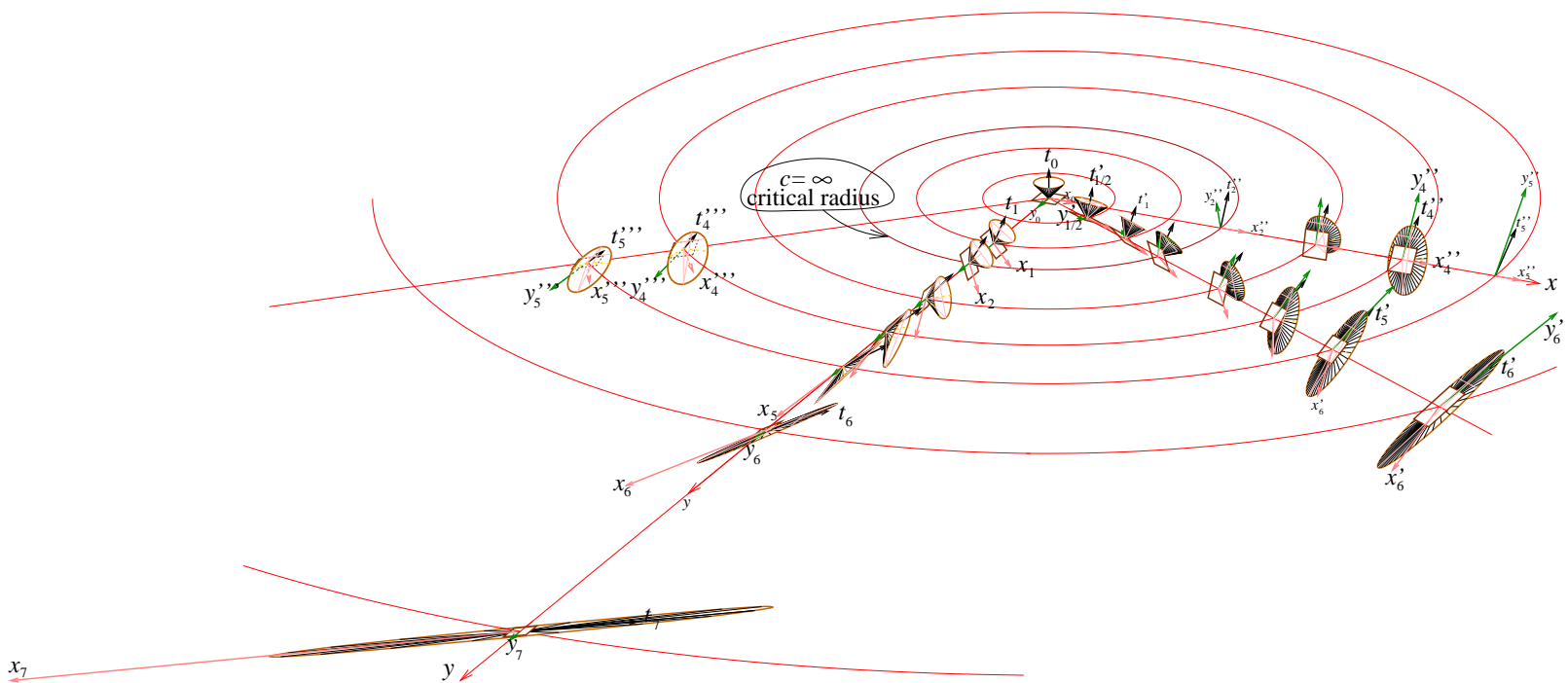
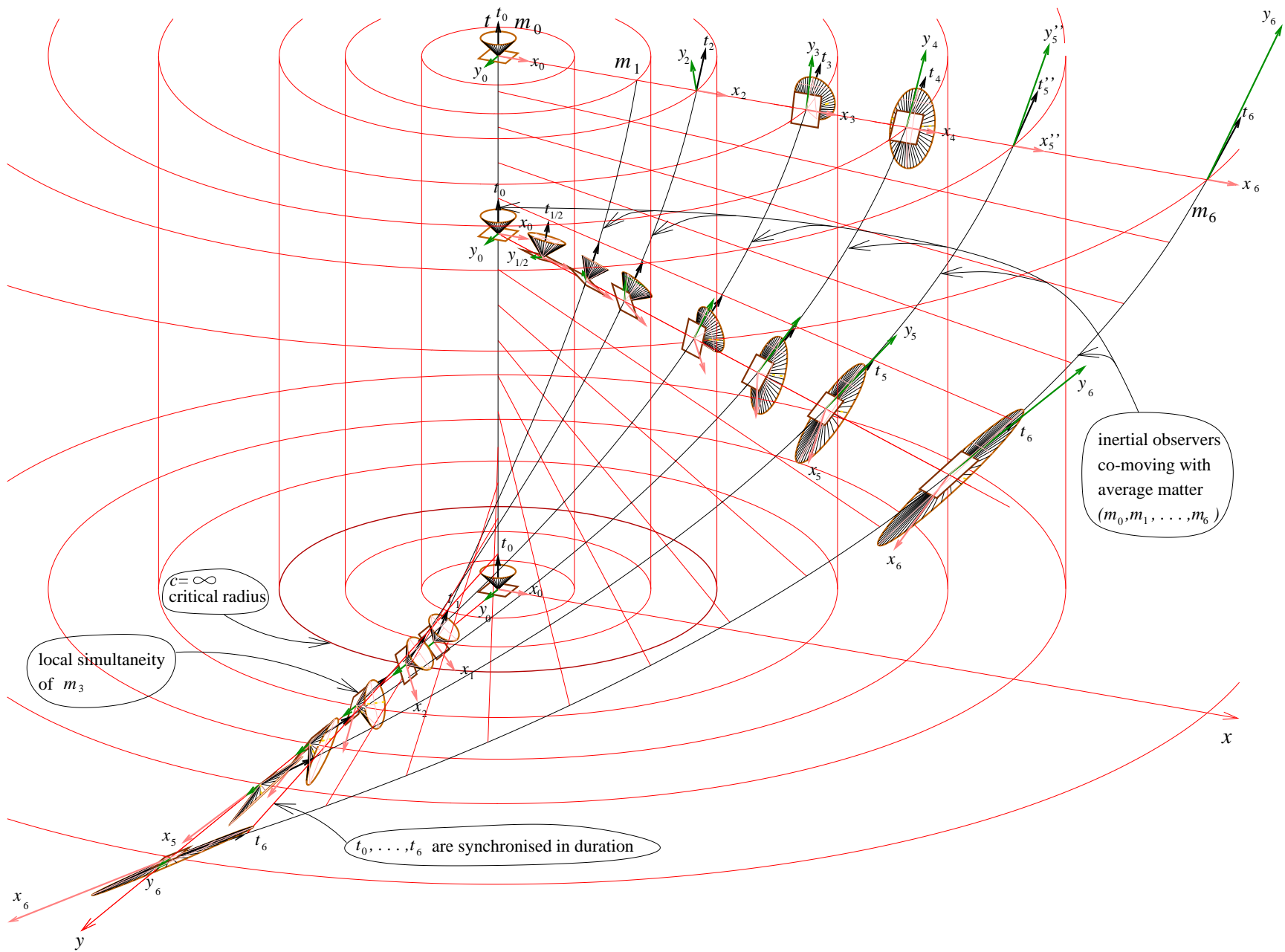


Figure 43: Tilted Spiral World, full view. Light-cones, life-lines, unit-vectors etc. Cf. Hawking-Ellis [18, Figure 61]. Cf. also Figure 1 herein.  $\omega \equiv \pi/30$ . Map 2 applies.



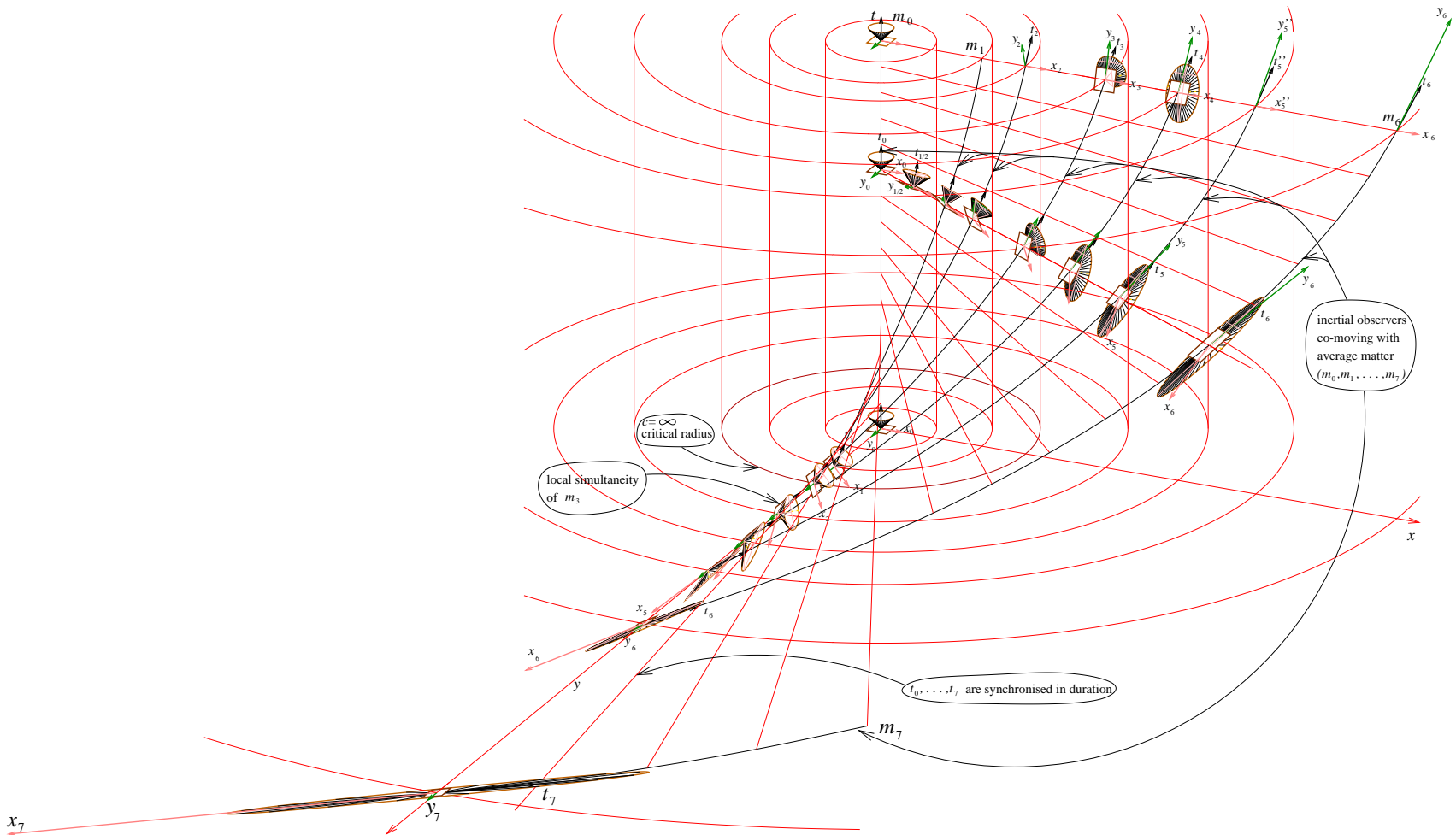


Figure 44: Tilted Spiral World, full view.  $\omega = \pi/30$ , Map 2 applies.

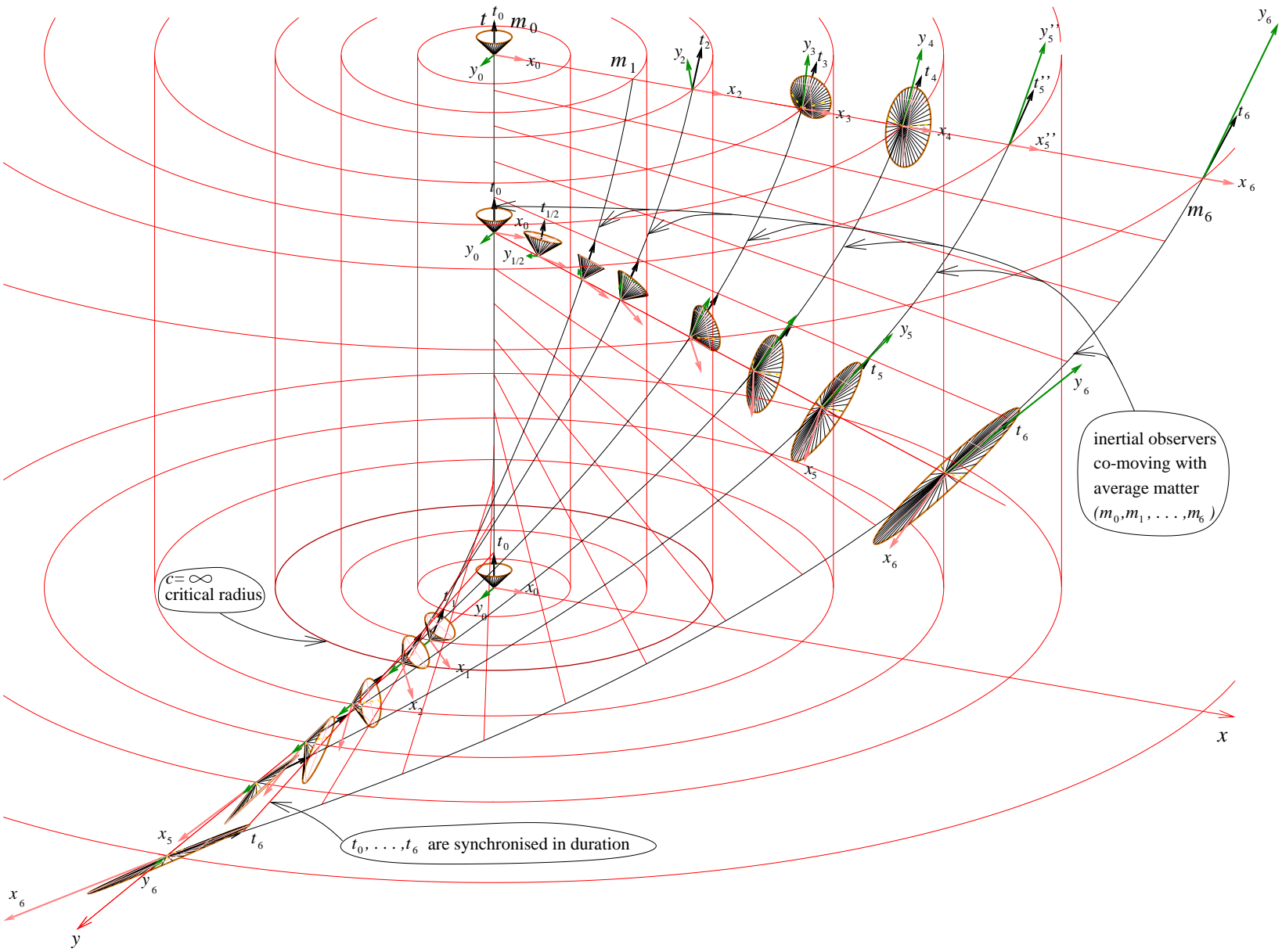


Figure 45: Full view of new spiral world. Cf. Hawking-Ellis [18], Figure 1 herein and the figure in Malament [25].  $\omega = \pi/30$ , Map 2 applies.

## 7 Giving physical meaning to cosmic compasses. What rotates in which direction (relative to whom).

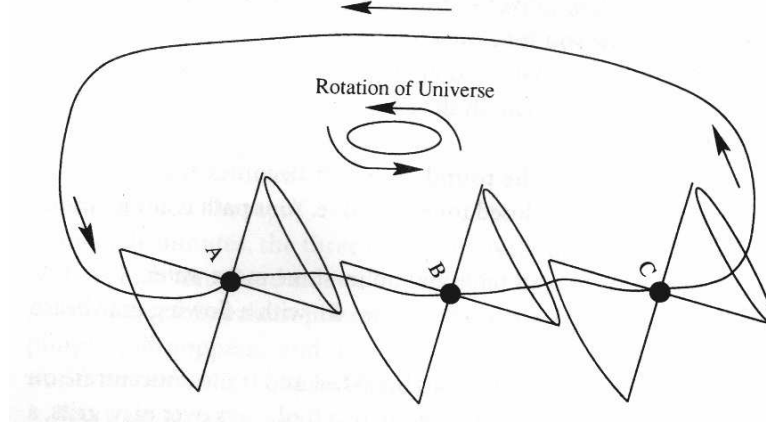


Figure 46: What rotates in which direction? The above is a picture from Pickover [31, p.185] from the chapter on Gödelian Universe implicitly offering a natural answer to this question. This is also Figure 7.5 in Gribbin [17, p.215].

In our “Tilted Spiral World” (Figures 42,43) the light cones are very strongly tilted forwards with increasing radius  $r$ . Therefore, if  $m_0$  throws a ball, say in the  $y$  direction, the ball will start moving in the  $y$  direction but with increasing radius it will *have* to turn in the  $\varphi$  direction because the life-line of the ball has to stay inside the light-cones (i.e. it has to be a time-like curve). The same applies even to a photon in place of the ball. This effect is called the *gravitational drag effect*<sup>13</sup> and is illustrated e.g. in our Figure 1 or equivalently in Figure 31 of Hawking-Ellis [18] as the curving of the photon-geodesics. The drag effect affects those and only those inertial bodies which are not at rest relative to one of the  $m_i$ ’s. This drag effect is present in the Naiv GU, too, but in a less dramatic way. To study the drag effect in our Tilted GU (in Figures 43, 36), we notice that our Tilted Dervish World (Figure 36) is structurally very close to Gödel’s original universe described and studied in Gödel [15], Hawking-Ellis [18, pp.168-170] and later papers. Hence the results about the drag effect in Gödel’s universe obtained in these works are applicable to our version of GU in Figure 36. The drag effect can be analyzed and described by studying the behavior of geodesics. Indeed, Figure 1 represents “dragging” of some characteristic geodesics. Let us be in dervish world. Then Figure 1 indicates the following. A ball thrown by  $m_0$  will start out radially, then will make a big circle and will come back to  $m_0$  from a new direction. From now on, we will call the circular motion or rotation traced out by this circle the *drag rotation*. In Figure 1 the direction of the drag rotation coincides with the  $\varphi$ -direction which in turn coincides with the direction of CTC’s. All this remains true in our Tilted Dervish World (Figure 36). In the Tilted Spiral World, matter (the  $m_i$ ’s) is seen to rotate in the same direction  $\varphi$ . Therefore in the Tilted Spiral World what we said above about the drag rotation, CTC’s etc. remains true. Hence, in the Tilted Spiral World the drag rotation is even stronger than in the dervish world and points in the same direction  $\varphi$  in which the matter content of the universe rotates. Hence

<sup>13</sup>What we call drag effect is often called *dragging of inertial frames*. For references on gravitational drag



in the Tilted Spiral World, we have an *increased drag effect*. As a curiosity we note that in the Tilted Spiral World everything rotates in the same direction  $\varphi$ .

Next we turn to replacing our cosmic compasses<sup>14</sup> with physically tangible compasses of an “observational” kind (i.e. subject to testing by thought experiment). In general relativity, the devices used for this purpose are called *gyroscopes* or *compasses of inertia*. The nonspecialist reader does not need to recall the definition, what we write below is amply enough for the present paper. The most important property (for us) of gyroscopes is that their working is based on inertial motion, hence the behavior of geodesics will also influence the behavior of gyroscopes. For the non-physicist reader we note the following.

In Newtonian physics it is provable that certain devices called gyroscopes preserve their directions despite of our moving them around, in other words, they behave like “cosmic compasses”.<sup>15</sup> We do not recall the definition of gyroscopes in detail. However we note that they can be made smaller and smaller in some sense such that their Newtonian property of preserving direction (whatever this means) remains true in general relativity (here the basic idea is that general relativity agrees with Newtonian mechanics for small enough speeds [with sufficient precision]). The essential idea behind gyroscopes is that a rigid body rotating fast enough tends to preserve its axis of rotation (in Newtonian physics). If we make the body small enough, then the tangential velocities of its parts will tend to zero. Hence the tangential velocities involved can be made small enough for the Newtonian approximation to be satisfactory.

It is natural to assume that the increased drag effect in Tilted GU described above will “drag” the gyroscopes, too, in the  $\varphi$  direction. Indeed, an analysis of the geodesics of Gödel’s universe in Lathrop-Teglas [23] suggests that this is so.

Our next goal is to find a new coordinatization  $C^+$  for our Tilted GU in which the gyroscope directions do not rotate.<sup>16</sup> One needs not regard this new coordinatization  $C^+$  superior in some sense to e.g. our Tilted Spiral World or more “real” than Tilted, instead,  $C^+$  is a coordinatization with some interesting and useful properties.  $C^+$  will be a (new) spiral world. We will call this new spiral world Refined (or Choice 2) Spiral World. After constructing  $C^+$ , it will be worthwhile to reconstruct the dervish world in such a form that the *new local frames* (i.e. “veils” or “hands” of the whirling dervishes) will be frames co-rotating with the gyroscopes. Then the local frames will be what are called *local inertial frames* in general relativity. A representation of the dervish world with these new local inertial frames represented as the “veils” of the dervishes will be called Refined (or Choice 2) Dervish World. The two tilted spiral worlds (Choices 1,2) and the two tilted dervish worlds (Choices 1,2) represent the same space-time in different coordinates.

In the Refined Dervish World all the mass-carrier observers  $m_i$  are at rest, they are evenly distributed and they are completely alike, yet their compasses of inertia are rotating. This violates Mach’s principle that the state of zero rotation of an inertial frame should coincide with the state of zero rotation with respect to the distribution of matter in the universe. For Mach’s principle see e.g. Barbour [3] and [4]. For more references on the drag effect and its connection with Mach’s principle see page 69.

Above (p.52) we recalled a picture from Pickover [31] because it “addresses” the question of what rotates in which direction. (E.g. does the universe rotate in the same direction as the time-travelers (CTC’s) do?) To make the question meaningful, one has to tell relative to what coordinate system is the question understood.<sup>17</sup> Of course, one would like to name

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<sup>14</sup>which were “abstract directions” so far

<sup>15</sup>See e.g. Epstein [10, p.128] for nice illustration.

<sup>16</sup>Below by gyroscopes we always mean gyroscopes of  $m_0$ .

<sup>17</sup>For the sake of the definition of Tilted Spiral World, this question is ill-posed.

an “observable” coordinate system for asking such a question. A possibility is to choose that coordinate system in which the gyroscopes do not rotate.<sup>18</sup> This is  $C^+$  of our Choice 2 Spiral World. We will see that in  $C^+$  the directions of the various rotations are essentially different from the ones in Pickover’s picture. If one looks at  $C^+$  without any preparation, then the directions of rotations appear as ad hoc, almost counter-intuitive. However, at least in our opinion, the train of thought outlined in this paper may provide an explanation for the arrangement of these directions. For more on this question of counter-rotation in the case of rotating (Kerr-Newman) black holes see [2].

Let us return to our goal of finding a coordinatization  $C^+$  of our Spiral World in which gyroscope directions do not rotate.<sup>19</sup> We have already observed that gyroscopes do rotate in our Tilted Spiral World (Figure 43). There are two equivalent ways for finding  $C^+$ :

(i) We analyze the rotation of gyroscopes as seen from the Tilted Dervish World, we observe that they rotate in the  $\varphi$ -direction. This means that in the spiral world gyroscopes rotate faster than the dervish world itself does (i.e. faster than  $\omega$ ). We choose the *refined spiral coordinates* to co-rotate with these gyroscopes. Hence the “gyroscope”-directions will be fixed when viewed from the Refined Spiral World as we wanted.

(ii) The following turns out to be equivalent with what we outlined in (i) above. Let us go back to Section 5 p.31, where we refined our Naive GU to get Tilted GU. There, on p.31, we found two possible choices (Choices 1,2) for the desired fine-tuning. Of the two, so far we took the simpler one, Choice 1. Choice 2 consists of tilting the light-cones in the dervish world *backwards* i.e. in a direction *opposite* to that of  $\varphi$  (in Choice 1 we tilted them forwards). What we claim here is that the result of choosing Choice 2 in Section 5 is equivalent with the result of the refinements outlined in item (i) above. This is the reason why we call our newest refined spiral and dervish worlds outlined in item (i) above *Choice 2* worlds as well as Refined worlds.

The new Choice 2 spiral and dervish worlds are illustrated and elaborated (constructed) in the figures below. A natural question comes up: If we had to refine our Choice 1 worlds because the drag effect made the gyroscope directions rotate, how do we know that the same problem will not come up in the new Choice 2 worlds? The answer is two-fold. (1) The extremely strong drag effect in Choice 1 Spiral World was caused by tilting the light-cones forwards extremely with increasing radius  $r$ . Cf. Figure 42 for this effect. Now, in our Choice 2 Spiral World the light-cones are not tilted forwards so much, actually recall that Choice 2 was obtained from Choice 1 by tilting light-cones backwards (relative to our naive GU). So, this very strong drag effect affecting even the gyroscopes need not arise (more precisely, need not be strong enough for affecting the gyroscopes). Indeed, as we said earlier, our dervish world is very close structurally to Gödel’s original space-time (GU). Therefore results about the original GU are applicable to our versions (calibrated slightly differently). Now, the results in Lathrop-Teglas [23] can be used to conclude that in our Choice 2 Spiral World gyroscope directions are fixed, i.e. they do not rotate. This can be seen by their characterization of geodesics in basically<sup>20</sup> Choice 2 Spiral World, as well as from their claim that Choice 2 Spiral coordinates are so called Fermi coordinates.

---

<sup>18</sup>Technically, we have Fermi coordinates in mind.

<sup>19</sup>This means that in  $C^+$ , gyroscopes of  $m_0$  preserve their directions (relative to the coordinate system).

<sup>20</sup>Our Choice 2 Spiral World is structurally very close to the coordinatization  $\langle t, r, \theta, z \rangle$  of GU given in

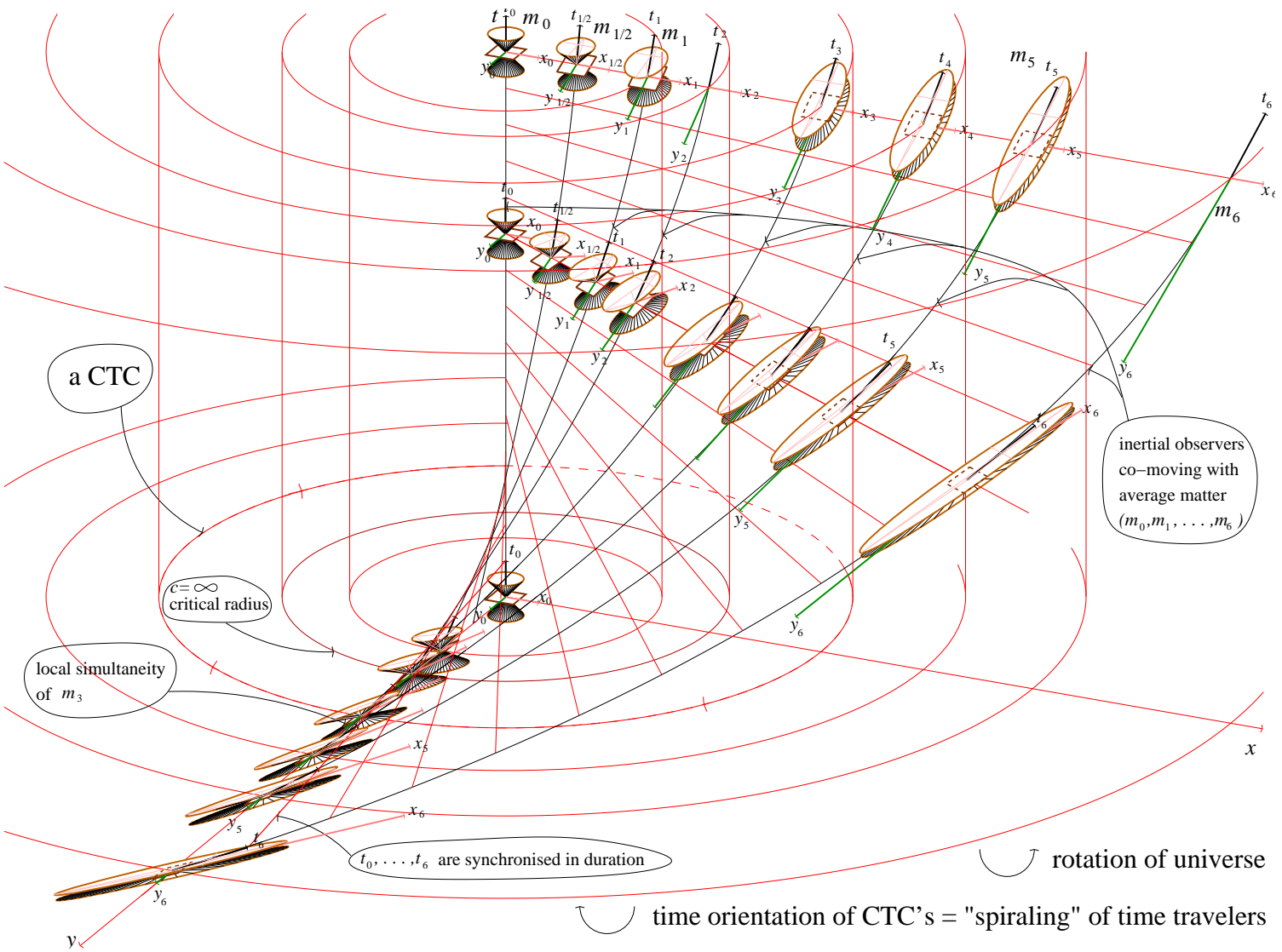


Figure 47: Choice 2 GU spiral view (i.e., Refined Spiral World). Here gyroscope directions are fixed (they do not change). (We are in Fermi coordinates in the sense of e.g. Lathrop-Teglas [23].)  $\omega = \pi/30$ , Map 2 applies.

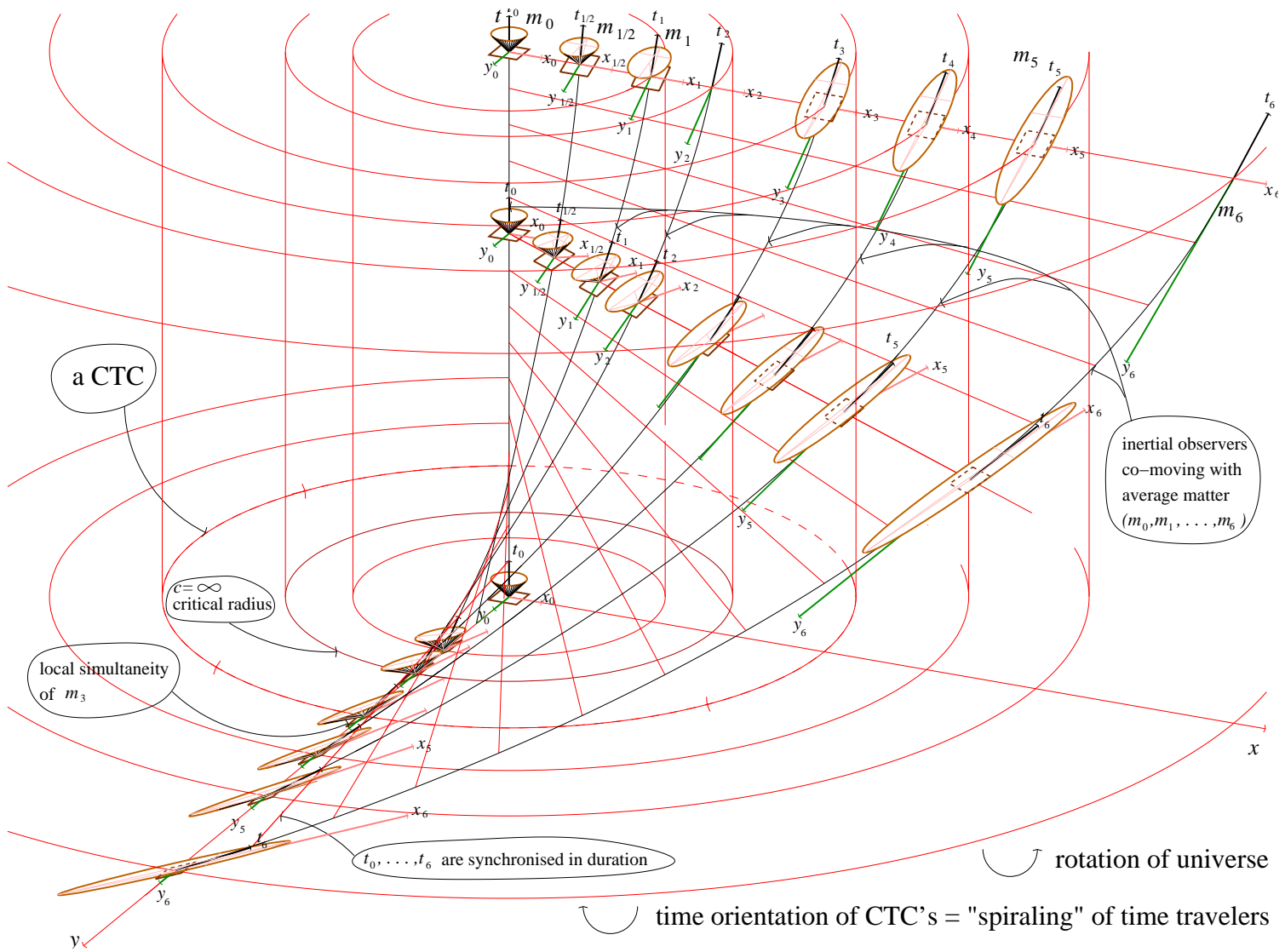


Figure 48: Choice 2 spiral view (Refined Spiral World). Here gyroscope directions are fixed (they do not change). (We are in Fermi coordinates in the sense of e.g. Latnrop-Teglas [23].)  $\omega = \pi/30$ , Map 2 applies.

Figure 49: Choice 2 spiral view. Cf. Fig.47 for more information.  $\omega = \pi/30$ , Map 2 applies.

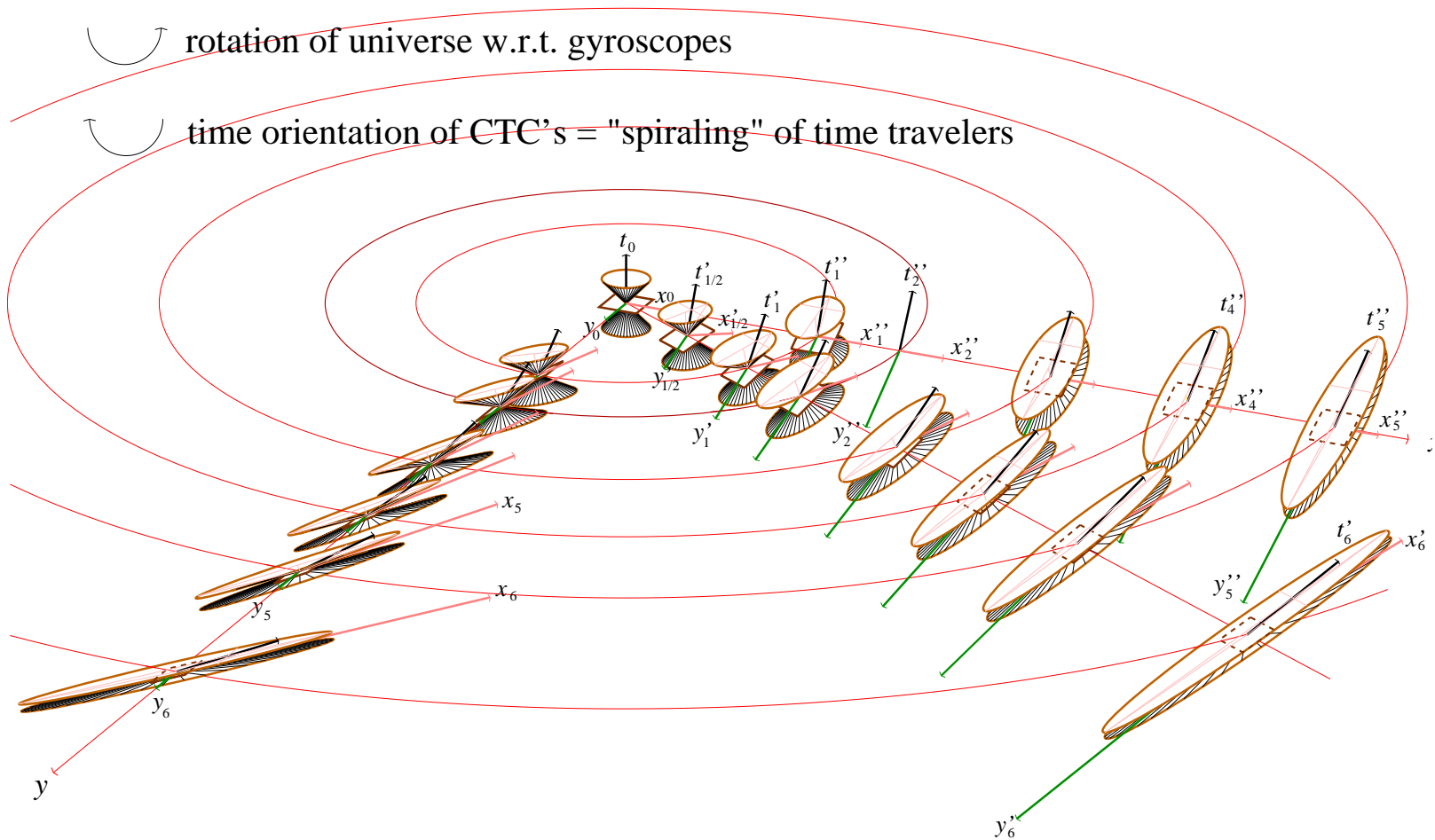
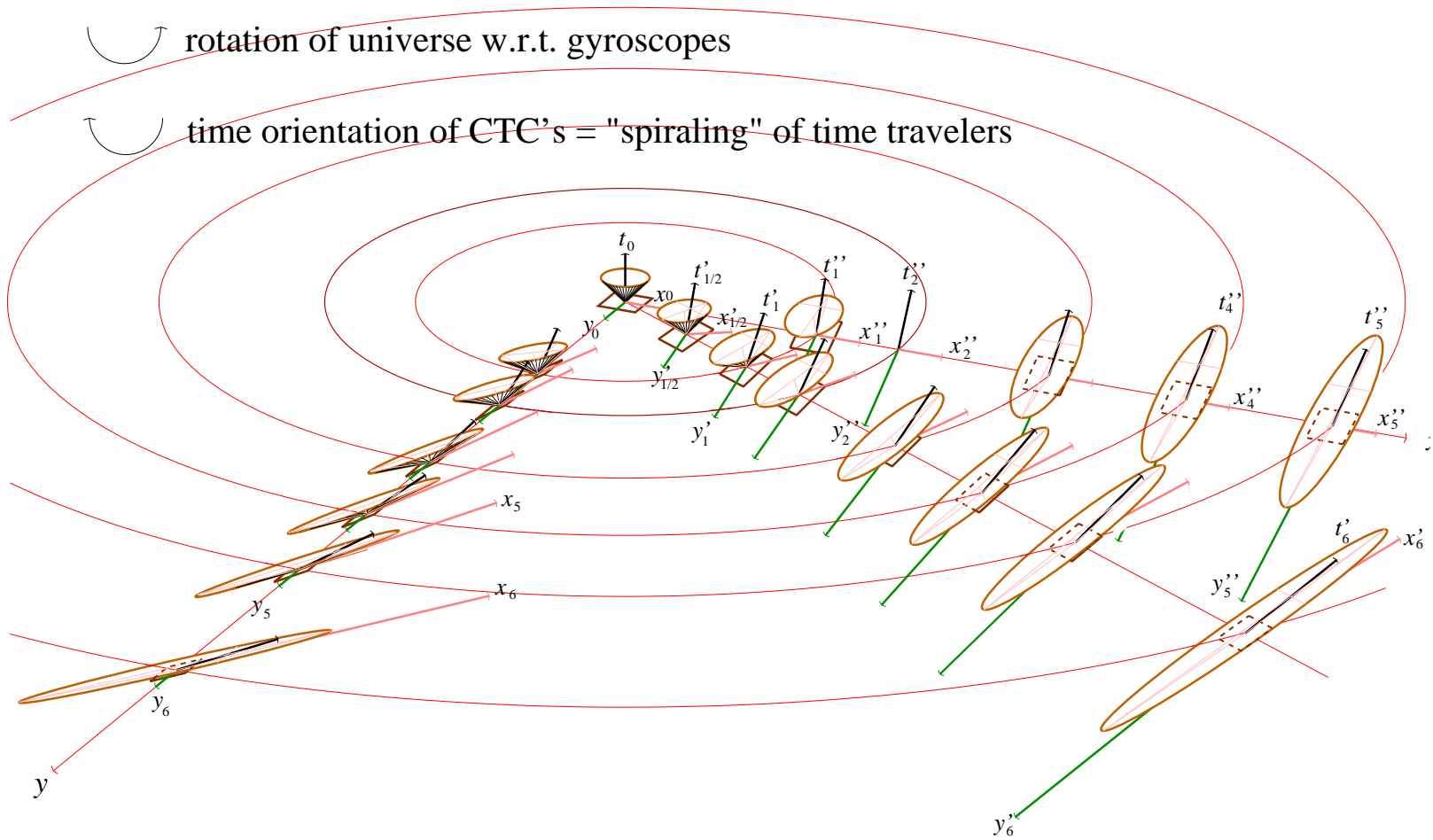


Figure 50: Choice 2 spiral view. Cf. Fig.48 for more information.  $\omega = \pi/30$ , Map 2 applies.



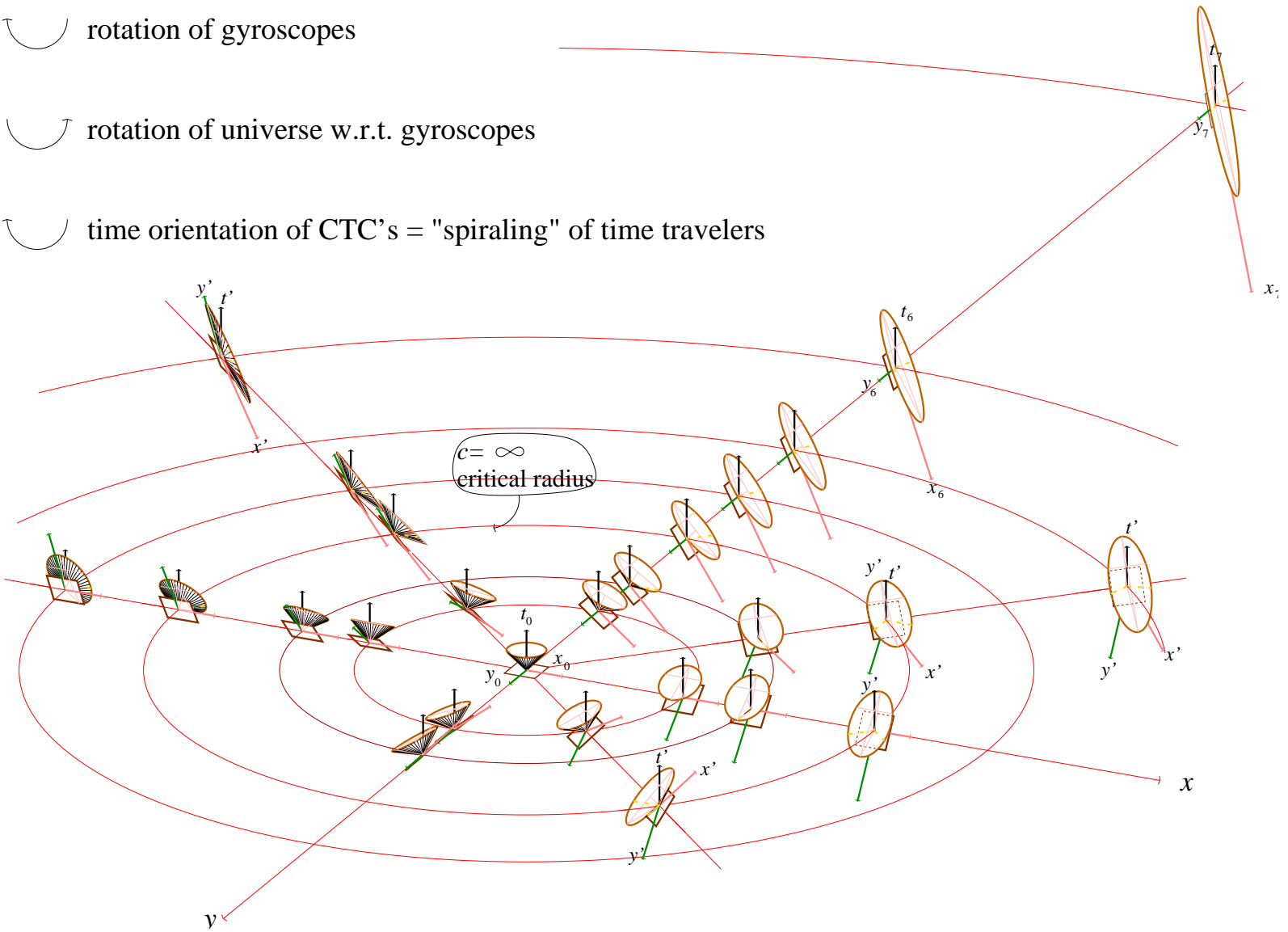
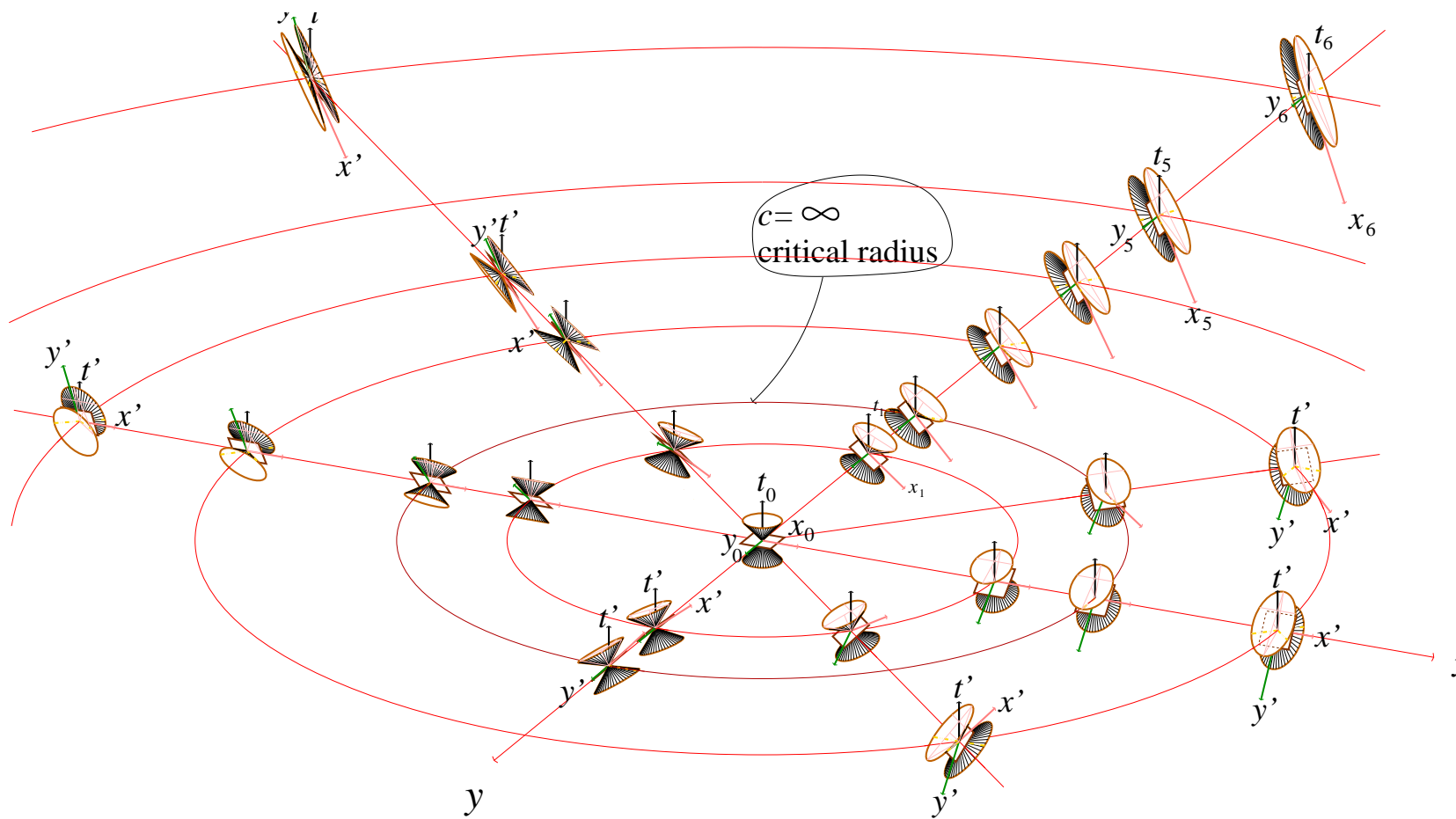




Figure 51: Dervish view in dual GTU (Choice 2). Compare with Figure 61 on p.169 in Hawking-Ellis [18] (cf. also Fig.1 herein).  $\omega = \pi/30$ , Map 2 applies.

Figure 52: Dervish view in dual GU (Choice 2).  $\omega = \pi/45$ , Map 1 applies.



 rotation of gyroscopes

 rotation of universe w.r.t. gyroscopes

 time orientation of CTC's = "spiraling" of time travelers



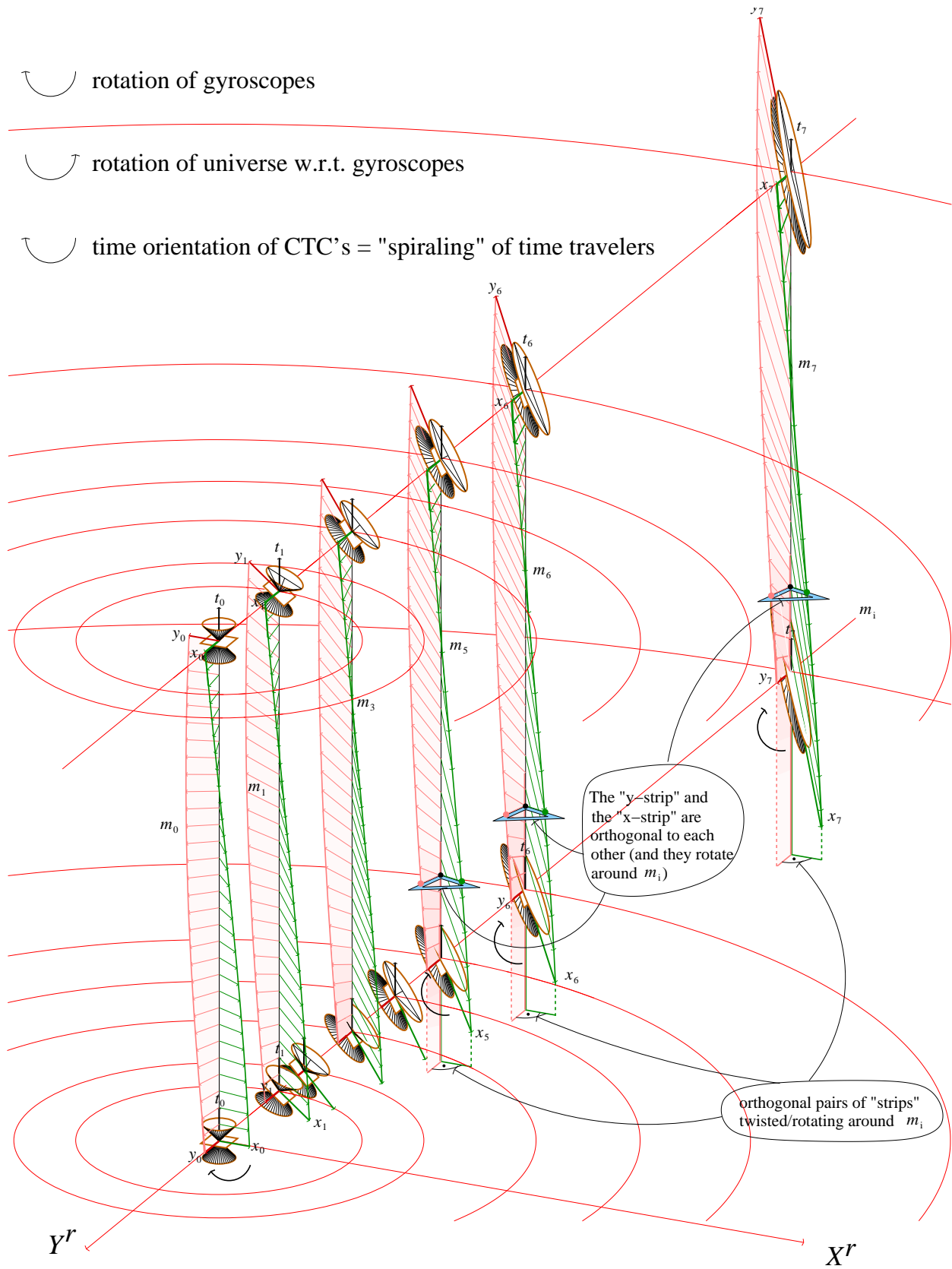




Figure 53: Tilted-dervishes, Choice 2 with original angular velocity.  $\omega = \pi/30$ , Map 2 applies.

 rotation of gyroscopes = "rotation of spiral world"

 rotation of universe w.r.t. gyroscopes

 time orientation of CTC's = "spiraling" of time travelers

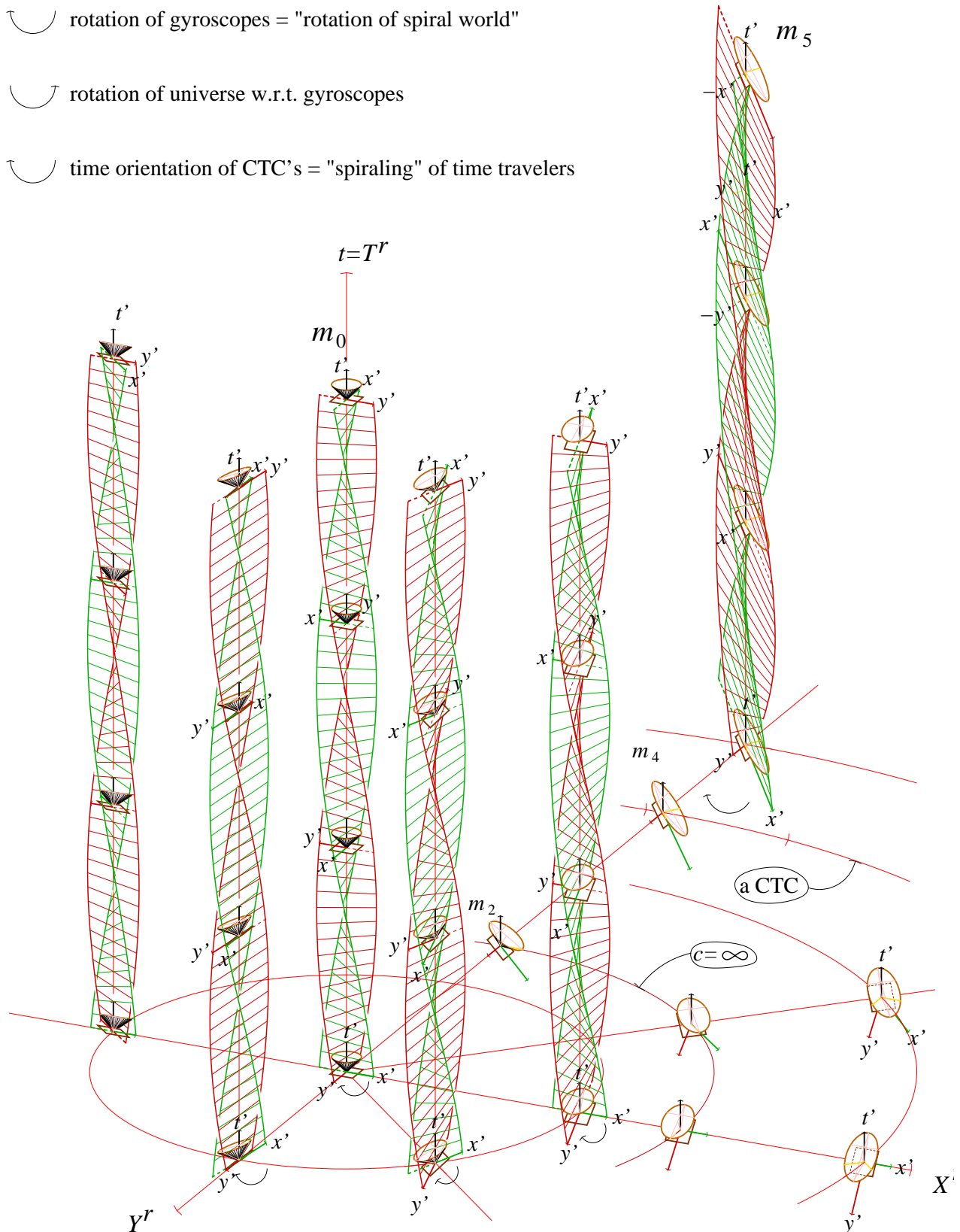




Figure 54: Dervish view in dual GU (Choice 2). " $\omega$  of universe" =  $\pi/60$  (recalibrated version of Map 2 applies). Spinning dervishes are artificially sped up ("artificial  $\omega$  of dervishes" =  $\pi/15$ ).

 rotation of gyroscopes = "rotation of spiral world"

 rotation of universe w.r.t. gyroscopes

 time orientation of CTC's = "spiraling" of time travelers

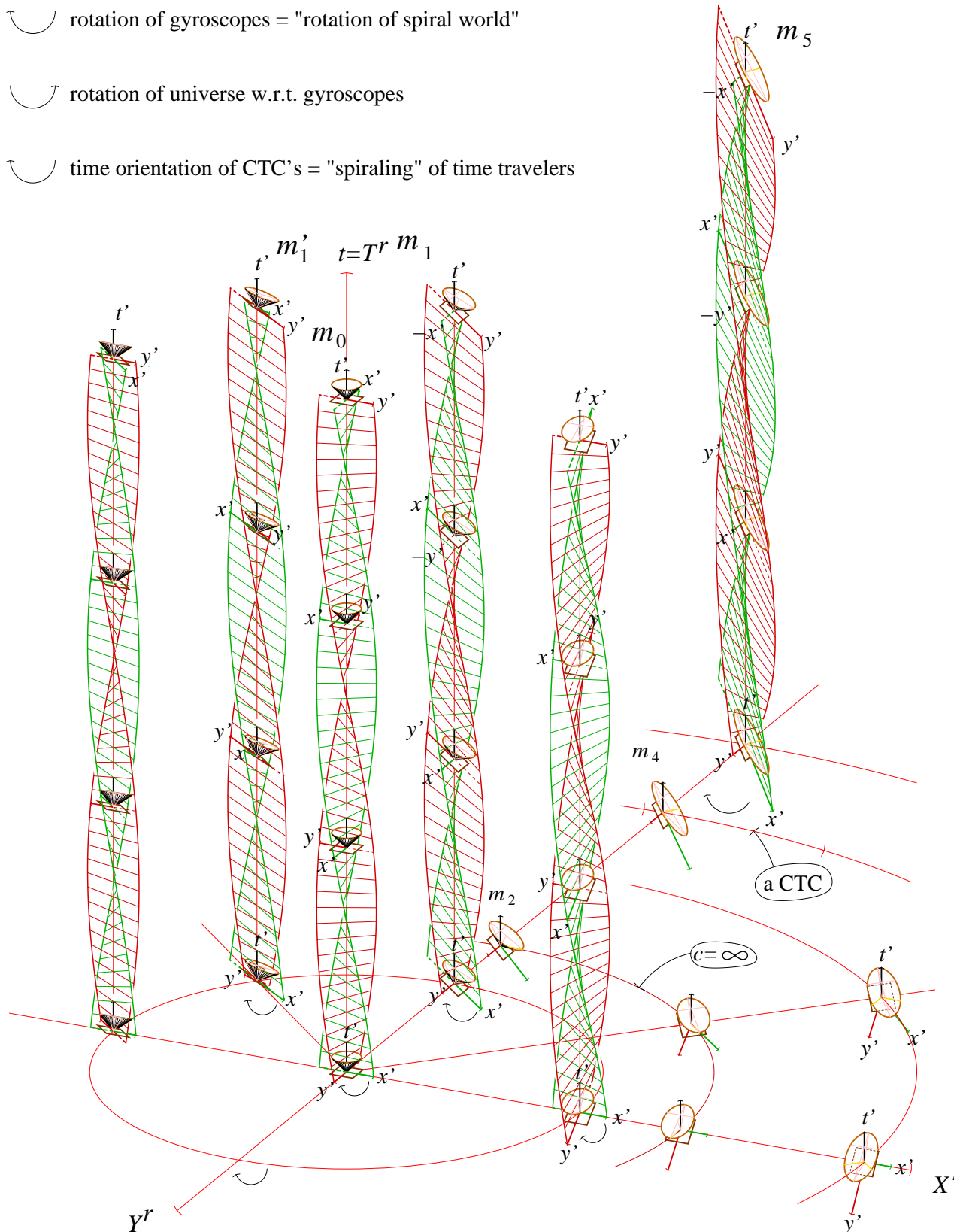


Figure 55: Dervish view in dual GU (Choice 2). " $\omega$  of universe" =  $\pi/60$  (recalibrated version of Map 2 applies). Spinning dervishes are artificially sped up ("artificial  $\omega$  of dervishes" =  $\pi/15$ ).

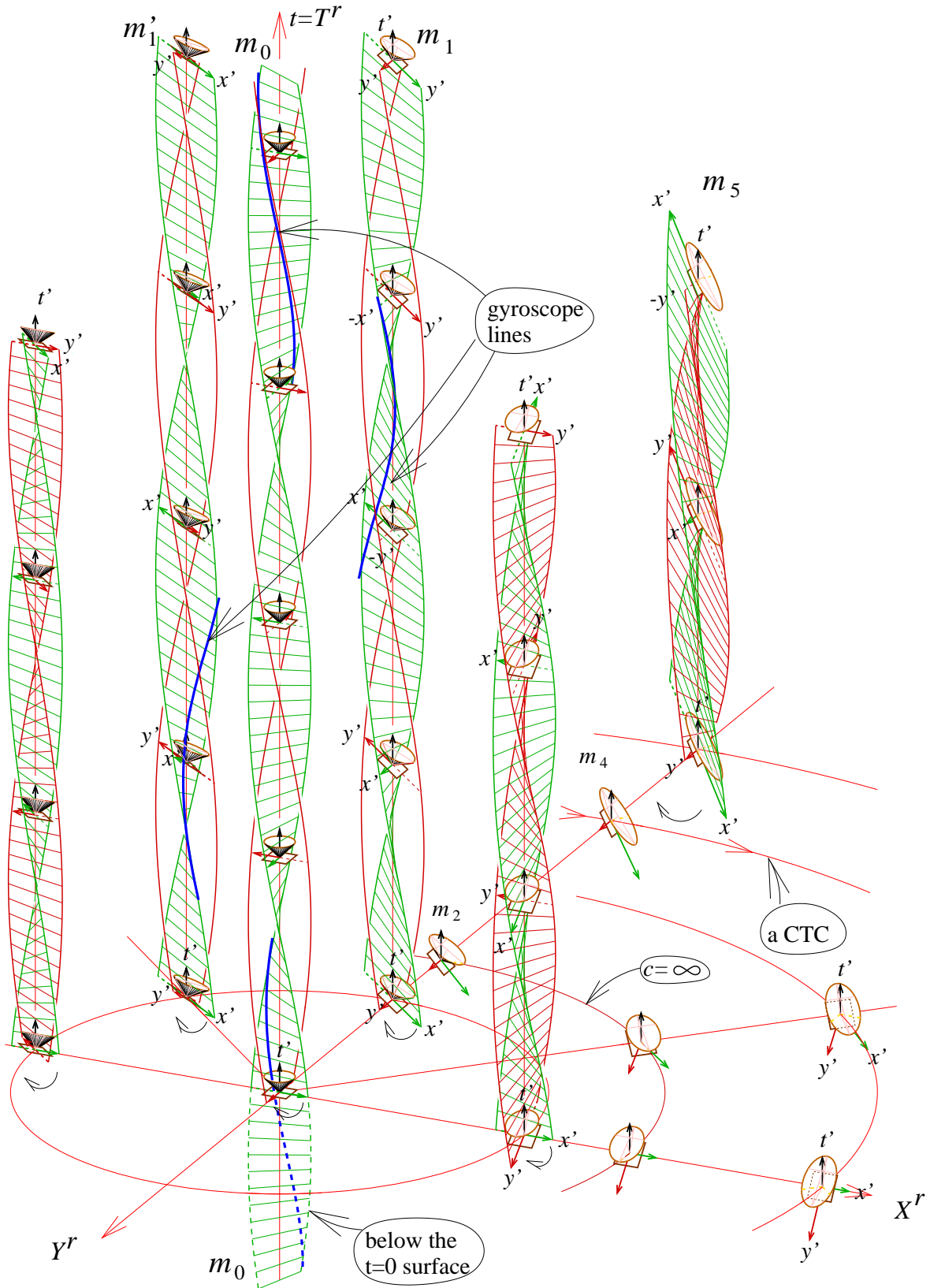


Figure 56: Choice 2 dervish view. “Fast” gyroscope lines. “ $\omega$  of universe” =  $\pi/60$  (recalibrated version of Map 2 applies). Spinning dervishes are artificially sped up (“artificial  $\omega$  of dervishes” =  $\pi/15$ ).

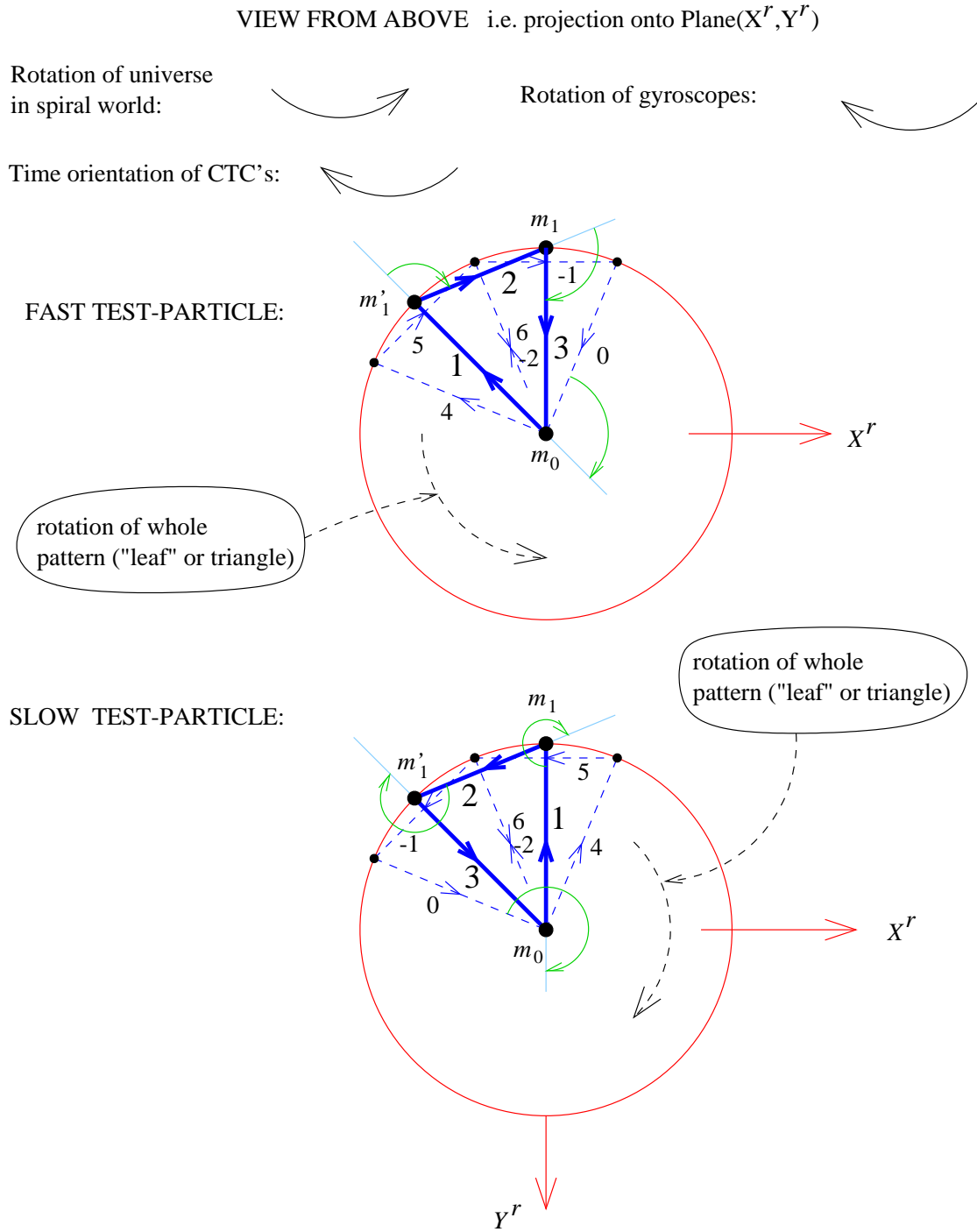


Figure 57: This belongs to the previous two figures involving gyroscope lines: Schematic paths of gyroscopes-directed test-particles. Such particles can be visualized as small spaceships whose pilots follow (the direction shown by) their gyroscopes strictly. Choice 2.

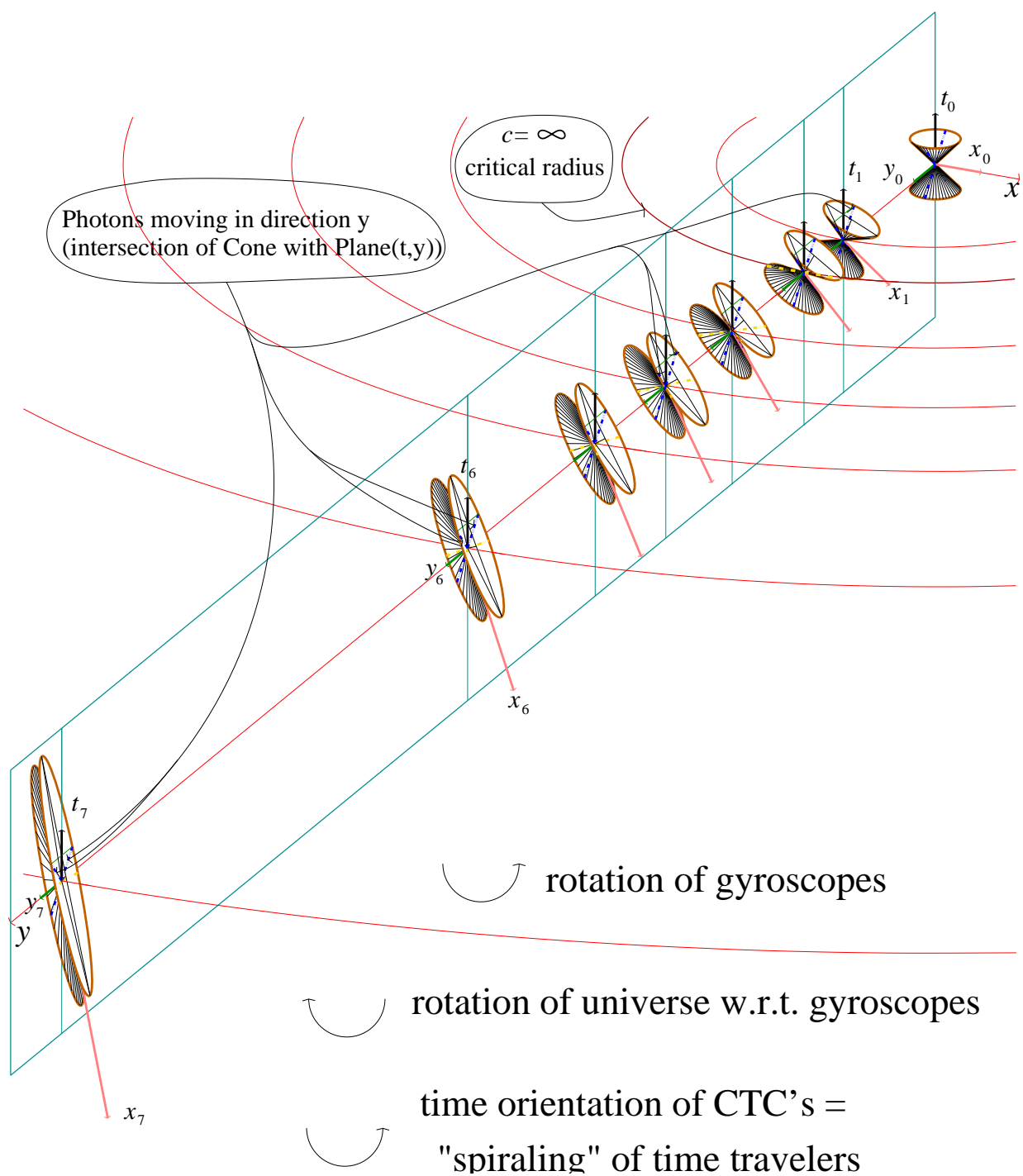


Figure 58: GU, Choice 2.  $\omega = \pi/30$ , Map 2 applies.

## 8 Metric tensors and some literature.

### 8.1 The metric tensor of the Naive GU.

The linear element in the Naive Spiral World is

$$ds^2 = -\frac{1-r^2\omega^2}{(1+r^2\omega^2)^2} dt^2 + dr^2 + dz^2 + \frac{r^2(1-r^2\omega^2)}{(1+r^2\omega^2)^2} d\varphi^2 - \frac{4r^2\omega}{(1+r^2\omega^2)^2} d\varphi dt.$$

Thus the components of the metric tensor  $\mathbf{g}$  of the Naive GU in the Naive Spiral World are

$$g_{tt} = -\frac{1-r^2\omega^2}{(1+r^2\omega^2)^2}, \quad g_{rr} = 1, \quad g_{zz} = 1, \quad g_{\varphi\varphi} = \frac{r^2(1-r^2\omega^2)}{(1+r^2\omega^2)^2}, \quad g_{\varphi t} = g_{t\varphi} = -\frac{2r^2\omega}{(1+r^2\omega^2)^2},$$

and the rest of the  $g_{ij}$ 's are 0. The nonzero Christoffel symbols  $\Gamma_{jk}^i$  are

$$\begin{aligned} \Gamma_{tt}^r &= \frac{r\omega^2(r^2\omega^2-3)}{(1+r^2\omega^2)^3}, & \Gamma_{tr}^t &= \frac{(1-r^2\omega^2)r\omega^2}{(1+r^2\omega^2)^2}, & \Gamma_{tr}^\varphi &= \frac{-2\omega}{(1+r^2\omega^2)^2 r}, \\ \Gamma_{t\varphi}^r &= \frac{2r\omega(1-r^2\omega^2)}{(1+r^2\omega^2)^3}, & \Gamma_{r\varphi}^t &= \frac{2r^3\omega^3}{(1+r^2\omega^2)^2}, & \Gamma_{r\varphi}^\varphi &= \frac{1-r^2\omega^2}{(1+r^2\omega^2)^2 r}, \\ \Gamma_{\varphi\varphi}^r &= \frac{r(3r^2\omega^2-1)}{(1+r^2\omega^2)^3}, & \text{and the } \Gamma_{kj}^i &= \Gamma_{jk}^i \text{ for the nonzero } \Gamma_{jk}^i \text{ listed above.} \end{aligned}$$

The scalar curvature is

$$R = 2\omega^2 \frac{(2r^2\omega^2-7)}{(r^2\omega^2+1)^2}.$$

Now,  $\Gamma_{rr} = \bar{0} = \langle 0, 0, 0, 0 \rangle$  shows that the radial straight lines in the  $xy$ -planes (i.e., the lines with direction “ $d\mathbf{r}$ ”) are geodesics. The life-lines of the galaxies are of direction  $\omega d\varphi + dt$ , hence

$$\omega^2 \Gamma_{\varphi\varphi} + 2\omega \Gamma_{\varphi t} + \Gamma_{tt} = \bar{0}$$

shows that the life-lines of the distinguished observers  $m_i$  are geodesics in the Naive GU.

Gödel wanted the distinguished observers  $m_0, \dots, m_i$  to be fully “equivalent” with each other. This means that  $m_i$  and  $m_0$  should be indistinguishable for any choice of  $m_i$ . This means that there should exist an automorphism  $h_{i,0} : \langle \mathbb{R}, \mathbf{g} \rangle \longrightarrow \langle \mathbb{R}, \mathbf{g} \rangle$  such that  $h_{i,0}$  takes the life-line of  $m_i$  to that of  $m_0$ . Since the scalar curvature is preserved by automorphisms, this implies that the scalar curvature should not depend on  $r$  (as it really does not depend on  $r$  in Gödel’s universe as we will see soon). This implies that in the Naive GU, the distinguished observers  $m_i$  are not fully equivalent with each other, because the scalar curvature depends on  $r$ .

We note that the linear element in the Naive Dervish World is

$$ds^2 = -dt^2 + dr^2 + dz^2 + \frac{r^2(1-r^2\omega^2)}{(1+r^2\omega^2)^2} d\varphi^2 + \frac{2r^2\omega}{(1+r^2\omega^2)} d\varphi dt.$$

## 8.2 The metric tensor of Gödel's universe GU.

Gödel in [15, p.275], [14, p.195] and elsewhere defines his universe by presenting the “linear element” (i.e. the “metric tensor field”) as

$$(\star) \quad ds^2 = \frac{2}{\omega^2}[-dt^2 + dr^2 + dz^2 + (\sinh^2 r - \sinh^4 r)d\varphi^2 + 2\sqrt{2}\sinh^2 r d\varphi dt].$$

This is understood in the *cylindric-polar coordinates*  $\langle t^d, r^d, \varphi^d, z^d \rangle$  of the dervish world we discussed in Sections 4,9. Cf. Figure 60. Instead of  $\frac{2}{\omega^2}$ , Gödel writes  $4a^2$  but in our notational system these two constants are basically the same. (One can interpret Gödel's  $a$  as  $a = \frac{1}{\sqrt{2}}\omega$ .<sup>21</sup> Anyway,  $a$  and  $\omega$  are only “parameters”.) Other differences are that Gödel used the  $+ - - -$  sign-convention and we also made a  $\varphi \rightarrow -\varphi$  coordinate transformation so as to use the same form of Gödel's metric that Lathrop-Teglas [23] uses. In tensorial form,  $(\star)$  can be written by specifying that Gödel's metric tensor field  $\frac{2}{\omega^2}\mathbf{g}$  is defined by

$$\begin{aligned} g_{tt} &= 1, & g_{rr} &= -1, & g_{\varphi\varphi} &= (\sinh^4 r - \sinh^2 r), & g_{\varphi t} &= \sqrt{2}\sinh^2 r, & g_{zz} &= -1, \\ g_{t\varphi} &= g_{\varphi t}, \text{ and the rest of the } g_{ij} \text{'s are } 0. \end{aligned}$$

Clearly,  $\mathbf{g}(p)$  is a function of  $p = \langle t, r, \varphi \rangle$ , but only  $\mathbf{g}_{\varphi\varphi}$  and  $\mathbf{g}_{\varphi t}$  depend on  $p$ . Further, of the parts of  $p$ , they depend only on  $r_p$  and on  $\varphi_p$ . This is caused by the symmetries of our space-time, i.e. rotation along  $\varphi$  and translation along  $t$  are automorphisms of GU (both for all versions of GU herein as well as in Gödel's quoted<sup>22</sup> papers). Notice that in the Naive Dervish World, both  $\mathbf{g}_{\varphi\varphi}$  and  $\mathbf{g}_{t\varphi}$  tend to constants as  $r$  tends to infinity while in Gödel's Dervish World they both tend to infinity as  $r$  tends to infinity. This is why we refined our Naive GU to obtain the Tilted GU.

Lathrop-Teglas [23] presents Gödel's universe in so-called Fermi coordinates. This means that the  $t$  axis as well as the radial lines are geodesics and the gyroscopes (i.e., compasses of inertia) of  $m_0$  are not rotating. This is a spiral world where the cosmic compasses are replaced with compasses of inertia. It is very similar to Refined (Choice 2) Spiral World depicted in Figure 48. Indeed, [23] obtains this metric from  $(\star)$  above by the following coordinate transformation. Below  $t', r', z', \varphi'$  are the new coordinates,  $t, r, z, \varphi$  are the coordinates used in  $(\star)$  and  $c = \frac{\sqrt{2}}{\omega}$ .

$$t' = ct, \quad r' = cr, \quad z' = cz, \quad \varphi' = \omega t' - \varphi.$$

This is the transformation from forward tilted (Choice 1) Dervish World to backward tilted (Choice 2) Spiral World (apart from multiplying with a constant  $c$ ). From now on, for simplicity, we write  $t, r, \varphi, z$  for  $t', r', \varphi', z'$ . Let us use the notation

$$\text{sh} = \sinh\left(\frac{\omega}{\sqrt{2}}r\right) \quad \text{and} \quad \text{ch} = \cosh\left(\frac{\omega}{\sqrt{2}}r\right).$$

Now, the “linear element” (i.e. the “metric tensor field”) of Gödel's universe in Fermi coordinates is

$$ds^2 = -(1 + 2\text{sh}^2\text{ch}^2)dt^2 + dr^2 + dz^2 + \frac{2}{\omega^2}\text{sh}^2(1 - \text{sh}^2)d\varphi^2 + \frac{4}{\omega}\text{sh}^4d\varphi dt.$$

<sup>21</sup>Cf. item (9) on p.191 in Gödel [14].

<sup>22</sup>The metric tensor field of Gödel's universe is a function of the coordinates  $(t, r, \varphi, z)$  of the world  $\langle t, r, \varphi, z \rangle$  [14, p.203].



The nonzero Christoffel symbols  $\Gamma_{jk}^i$  are

$$\begin{aligned}\Gamma_{tt}^r &= \omega\sqrt{2}\text{shch}((2\text{ch}^2 - 1)), & \Gamma_{tr}^t &= \omega\sqrt{2}\text{shch}, & \Gamma_{tr}^\varphi &= \omega^2\sqrt{2}\text{shch}, \\ \Gamma_{t\varphi}^r &= -2\sqrt{2}\text{sh}^3\text{ch}, & \Gamma_{r\varphi}^t &= \frac{\sqrt{2}\text{sh}^3}{\text{ch}}, & \Gamma_{r\varphi}^\varphi &= \frac{-\omega(2\text{ch}^4 - 4\text{ch}^2 + 1)}{\sqrt{2}\text{shch}}, \\ \Gamma_{\varphi\varphi}^r &= \frac{\sqrt{2}\text{shch}(2\text{ch}^2 - 3)}{\omega}, & \text{and the } \Gamma_{kj}^i &= \Gamma_{jk}^i \text{ for the nonzero } \Gamma_{jk}^i \text{ listed above.}\end{aligned}$$

The scalar curvature is

$$R = 2\omega^2.$$

A sample of papers investigating Gödel's universe is Chakrabarti-Geroch-Liang [5], Chandrasekhar-Wright [6], Dorato [8], Gödel [13], [15], Heckmann-Schücking [19], Kundt [22], Lathrop-Teglas [23], Malament [25], Obukhov [29], Plaue-Scherfner-de Sousa [32], Sklar [34], Stein [35]. A sample of books about general relativity and time (especially relevant to the present paper) is Earman [9], Gibilisco [12], Gott [16], Horwich [21], Novikov [28], O'Neil [30], Pickover [31], Yourgrau [40].

For more on the drag effect and its connections with Mach's principle cf. e.g. Wald [39, p.89 item 3.(c), p.187 Problem 3(b), p.319 immediately below item (12.3.17)]. For more detail on "drag" and Mach cf. Misner-Thorne-Wheeler [26, §21.12 (entitled "Mach's...") and especially pp.546-548, also item B on p.879, pp.1117, 699, 893, 1120]. Cf. also d'Inverno [7, §9.2 (pp.121-124)], Gibilisco [12, pp.19-123 (subtitle: Alone in the universe)]. Cf. also [26, pp.880-1] for nice drawings of rotating black holes.

For the gravitational drag effect we refer to Rindler [33, pp.10-13, §§1.15, 1.16], Wald [39, pp.9,71,89,183,319], Wald [38, pp.32-33], together with Misner-Thorne-Wheeler [40.7 (pp.1117-1120), §33.4 (p.892), §21.12 (in particular p.547), p.1120 (footnote)] MTW. The gravitational drag effect is related to Mach's principle as is explained e.g. in [26, §21.12] and in [33, §1.15 (e.g. p.12)].

Figure 1 is a slightly corrected version of Figure 31 in Hawking-Ellis [18]. This picture can also be found in Yourgrau [40]. Malament [25, p.99] pointed out that the light-cones on that figure are tilted so much that they do not contain the vertical lines which are the life-lines of the distinguished observers in the dervish-world (which the figure represents). Below we include the Figure from Malament's paper (in which the light-cones are corrected already).

The present work is part of a broader effort for what we could bluntly call demystifying general relativity theory and its relatives like wormhole-theory and cosmology. More concretely, we try to provide a purely logic based conceptual analysis for general relativity and its relatives. One of the aims is to provide a technically correct but easily understandable introduction to general relativity including its most exotic reaches for the questioning mind of the nonspecialist. A sample of works in this general direction is [1], [2], [24], [36], [37].

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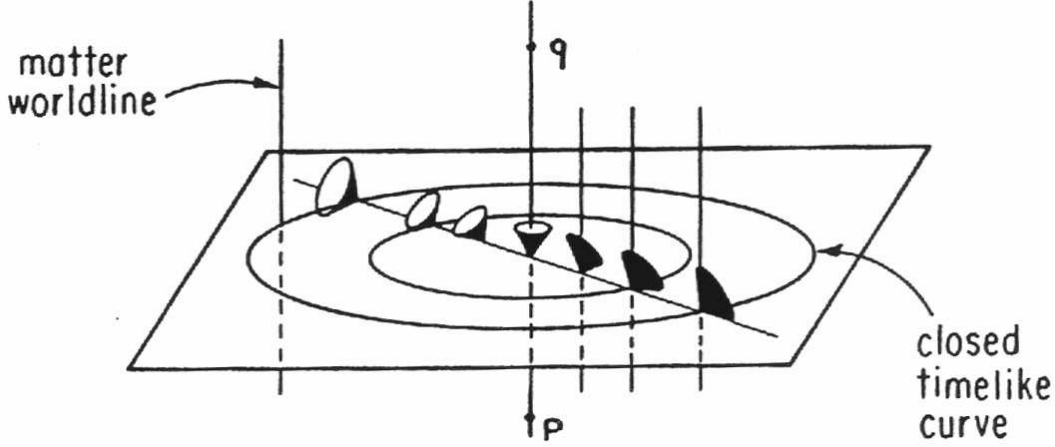


Figure 59: Figure from Malament's paper [25].

## 9 Appendix: technical details for the constructions.

**Connections between our spiral coordinate system  $\langle t, x, y, z \rangle = \langle t^s, \dots, z^s \rangle$  and co-rotating (dervish) coordinate system  $\langle t', x', y', z' \rangle = \langle T^r, X^r, Y^r, Z^r \rangle$ :**

By definition,  $t' = t$  and  $z' = z$ . Throughout we suppress the irrelevant spatial coordinate  $z$ . Below, instead of the Cartesian systems  $\langle t, \dots, y \rangle, \langle t', \dots, y' \rangle$  we use their cylindric-polar-coordinates variants  $\langle t, \varphi, r \rangle$  and  $\langle t', \varphi', r' \rangle$ .<sup>23</sup> The connections are the usual standard ones, e.g.  $r = \sqrt{x^2 + y^2}$ ,  $y = r \cdot \cos(\varphi)$ ,  $x = r \cdot \sin(\varphi)$ ,  $\varphi = \arctan(x/y)$ . In more detail,  $r(p) = \sqrt{x(p)^2 + y(p)^2}$  etc.  $\langle t^s, \varphi^s, r^s \rangle := \langle t, \varphi, r \rangle$  and  $\langle T^r, \varphi^r, r^r \rangle = \langle t^{der}, \varphi^{der}, r^{der} \rangle = \langle t', \varphi', r' \rangle$ . Here  $s$  abbreviates “spiral” and “der” abbreviates “dervish”.

The “galaxies”  $m_1, m_2, \dots, m_i$  appear as rotating around  $m_0$  in direction  $\varphi$  with angular velocity  $\omega$  in  $\langle t^s, \dots \rangle$  while their cosmic compasses  $x_i, y_i$  appear fixed (non rotating). As a contrast,  $\langle T^r, \dots \rangle$  shows  $m_1, \dots, m_i$  as motionless, while it shows their cosmic compasses as rotating in direction  $-\varphi$  with angular velocity  $\omega$ . We use  $p$  to denote an arbitrary point which has coordinates  $t(p), \varphi(p), r(p)$  etc. We represent these simple connections in Figures 60–62. As we said, we suppress coordinate  $z$ . In Figure 60 below (p.71) we regarded only such points  $p$  which are on the cylinder  $r(p) = 1$ . Generalizing to arbitrary points is trivial since  $r$  does not change. As it is obvious from the picture, the transformation “spiral”  $\mapsto$  “dervish” is

$$\varphi^d(p) = \varphi^s(p) - \omega \cdot t^s(p)$$

$$r^d(p) = r^s(p)$$

$$t^d(p) = t^s(p)$$

$$z^d(p) = z^s(p). \text{ Clearly,}$$

$$\varphi^s(p) = \varphi^d(p) + \omega \cdot t^d(p).$$

The angular velocity of the rotation of the universe as seen by  $\langle t^s, \dots \rangle$  is  $\omega$ .

<sup>23</sup>Cf. [25, Fig. 10.2 (p. 252)].

The diagram illustrates the relationship between different coordinate systems in cosmology, showing how the direction of rotation of distant galaxies differs from the expected rotation of local cosmic compasses.

**View from the spiral coordinate system  $\langle t^s, \varphi^s, r^s \rangle$ :**

- The vertical axis is labeled  $t^s = t^d$ .
- A horizontal plane is labeled "the  $\varphi^s = 0$  plane".
- A box indicates  $\varphi^d = 0$ !
- A point  $p$  is shown on the life-line of  $m_i$  (a galaxy).
- The life-line of  $m_i$  is a blue curve.
- The world-line of  $p$  is a red curve.
- The direction of rotation of distant galaxies is indicated by  $\omega$ .
- The direction of rotation of universe (i.e. of distant galaxies) w.r.t. cosmic compasses i.e. in  $\langle t^s, \varphi^s, r^s \rangle$  is indicated by  $\varphi^d(p)$ .
- The equation  $\varphi^d(p) = \varphi^s(p) - \omega \cdot t^s(p)$  relates the two directions of rotation.
- The equation  $t^d(p) = t^s(p)$  shows that time is the same in both systems.
- The equation  $\varphi^d(p) = \varphi^s(p) - \omega \cdot t^s(p)$  shows that the angular position of the cosmic compass is shifted relative to the spiral coordinate system.

Figure 60: As throughout this work, here too, the irrelevant spatial coordinates  $z^{\text{d}} = z^{\text{s}} = z_{\text{i}} = z$  are suppressed.

View from the dervish coordinate system  $\langle t^d, \varphi^d, r^d \rangle$ :

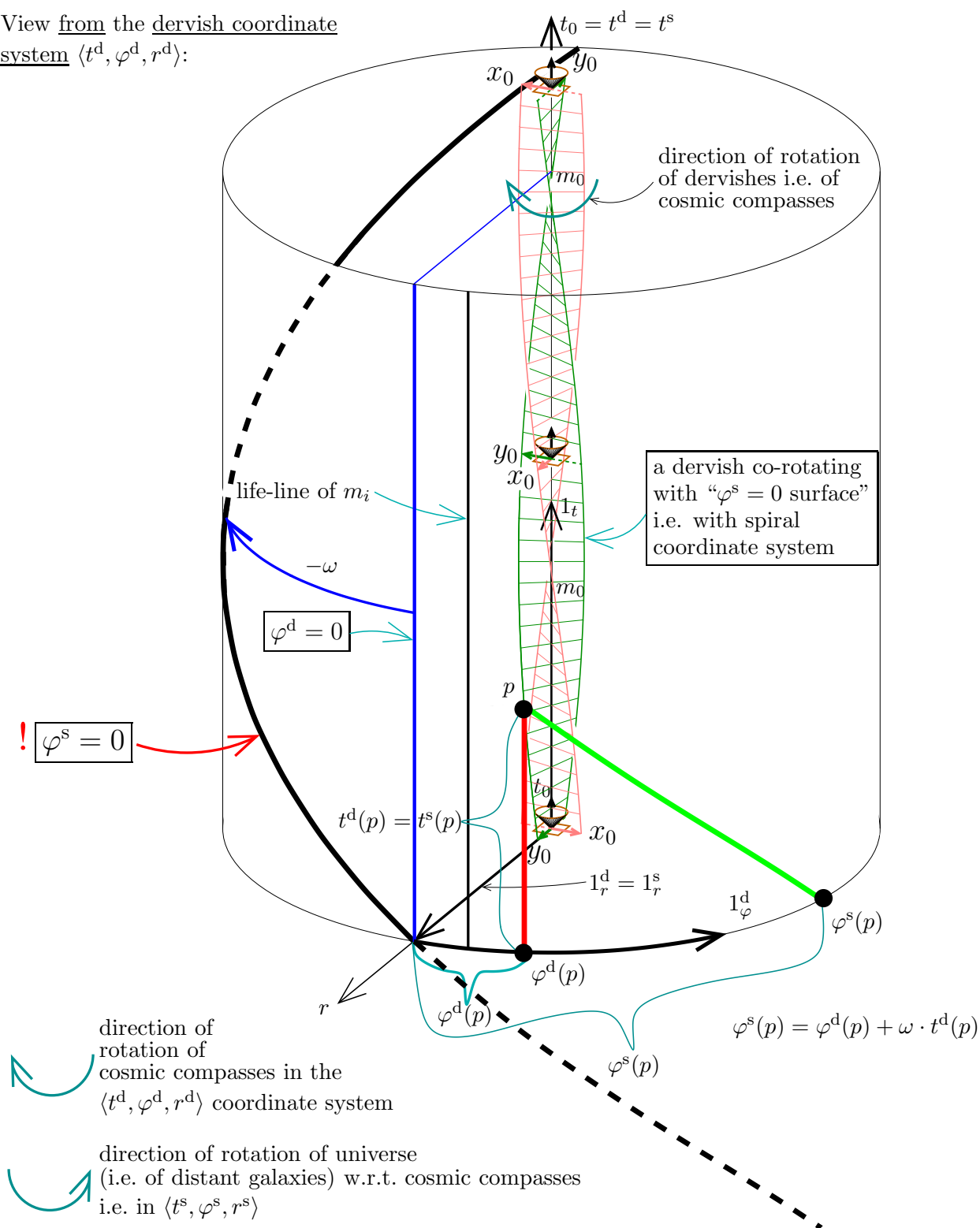
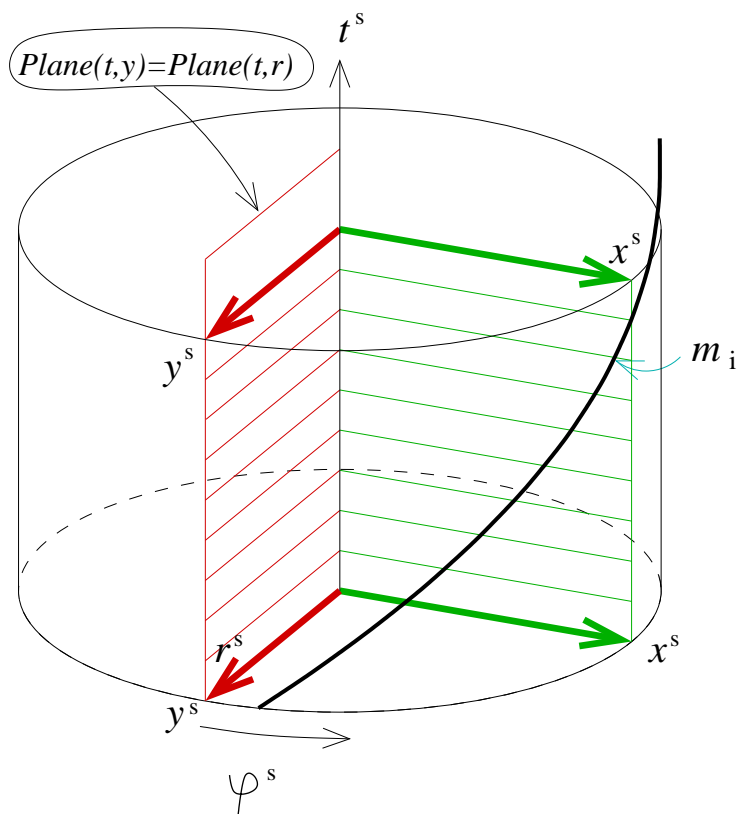
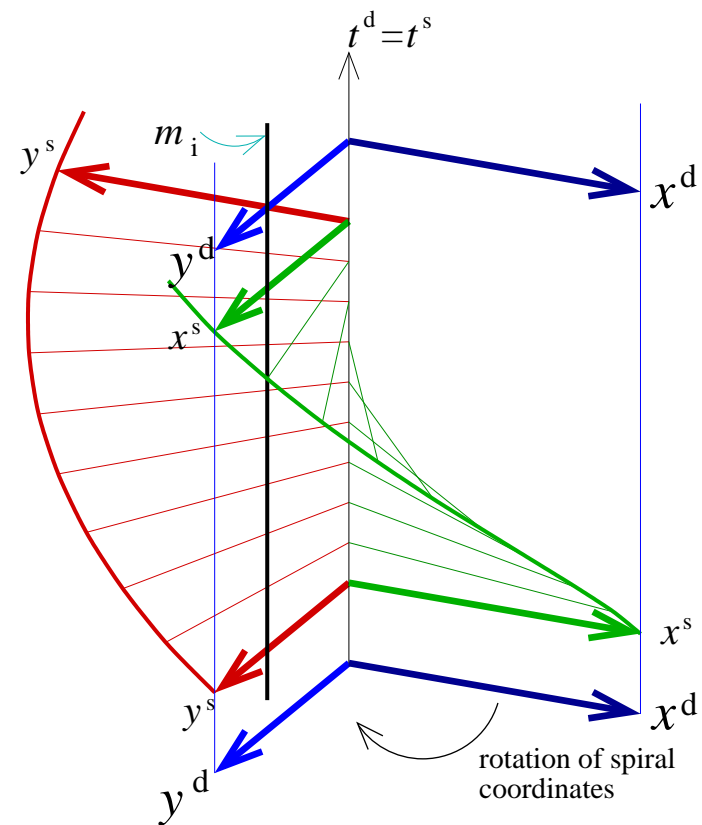


Figure 61: Dervish view of spiral world, i.e. backward transformation  $\langle t^{\text{der}}, \dots \rangle \longrightarrow \langle t^{\text{spi}}, \dots \rangle$ . Notice that the  $t = 0$  plane in this figure coincides with that of previous figure (e.g. marked points are the same on the two).

SPIRAL VIEW:



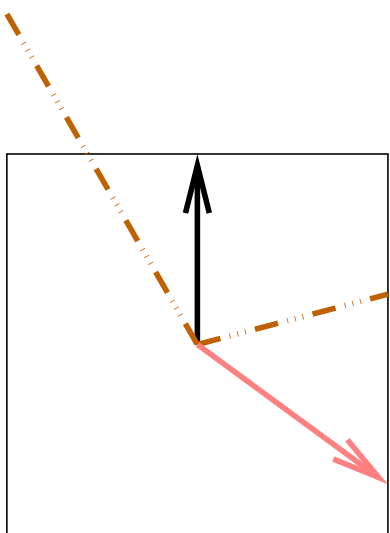
DERVISH VIEW:



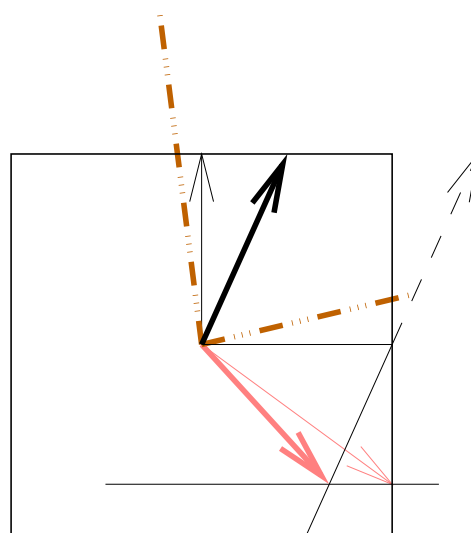
rotation of cosmic compasses  
seen in  $\langle T^I, \dots \rangle = \langle t^{der}, \dots \rangle = \langle t^d, \dots \rangle$

rotation of universe =  
"rotation" of galaxies w.r.t cosmic compasses

Figure 62:

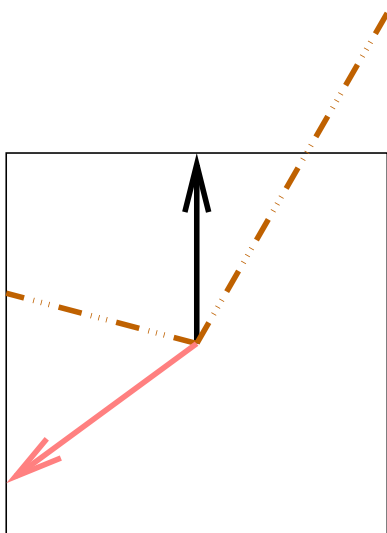


Godel dervish, Choice 1

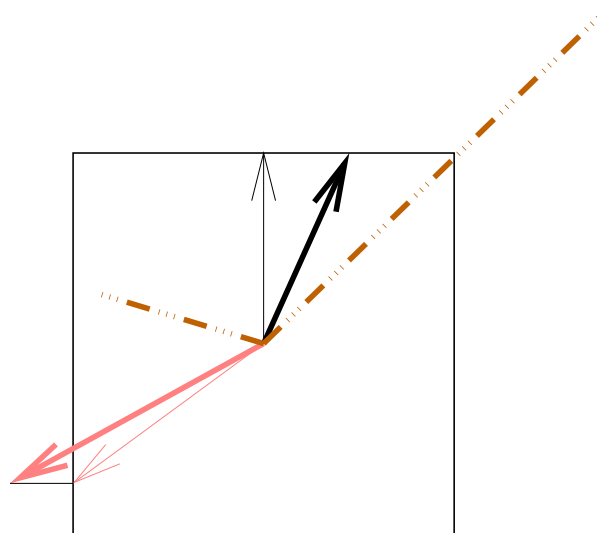


new spiral, Choice 1

$v=0.4501527$   
observer  $m_1$

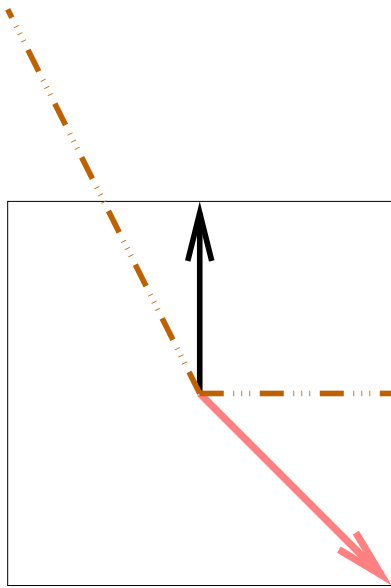


Godel dervish, Choice 2

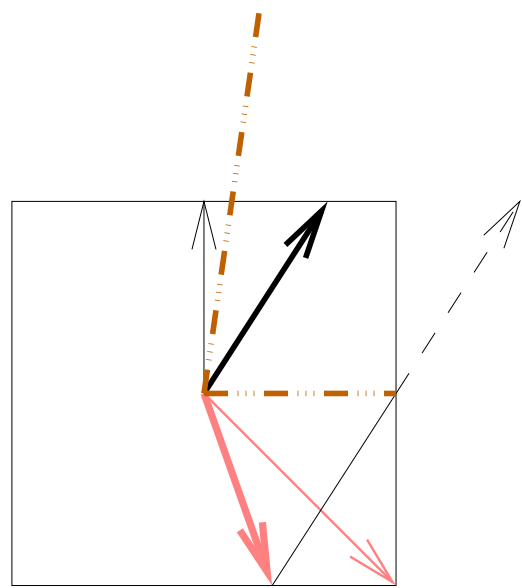


new spiral, Choice 2

Figure 63: Details of observer  $m_1$ .



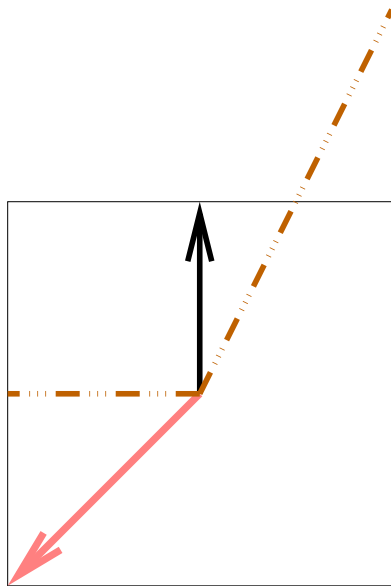
Godel dervish, Choice 1



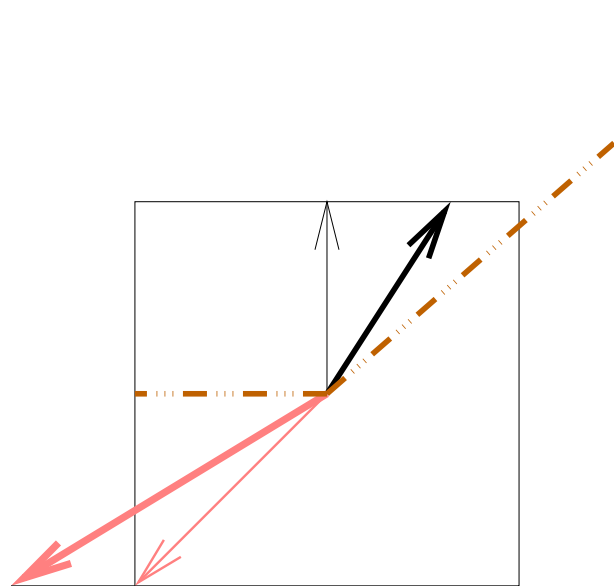
new spiral, Choice 1

$v=0.6447222$

observer  $m_2$

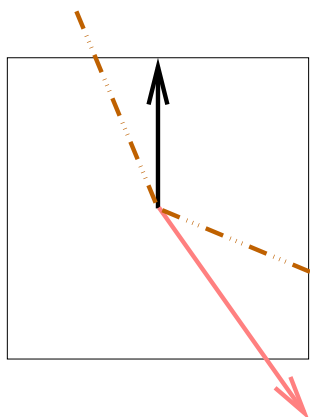


Godel dervish, Choice 2

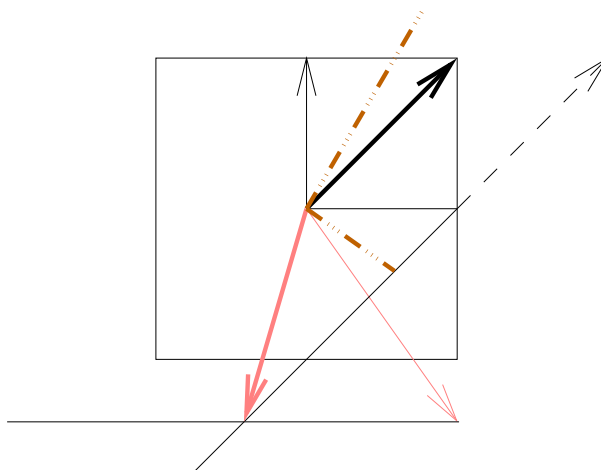


new spiral, Choice 2

Figure 64: Details of observer  $m_2$ .

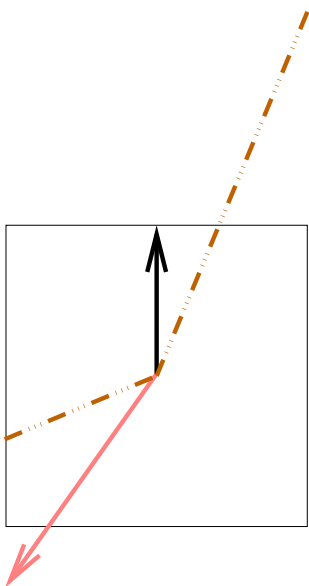


Godel dervish, Choice 1

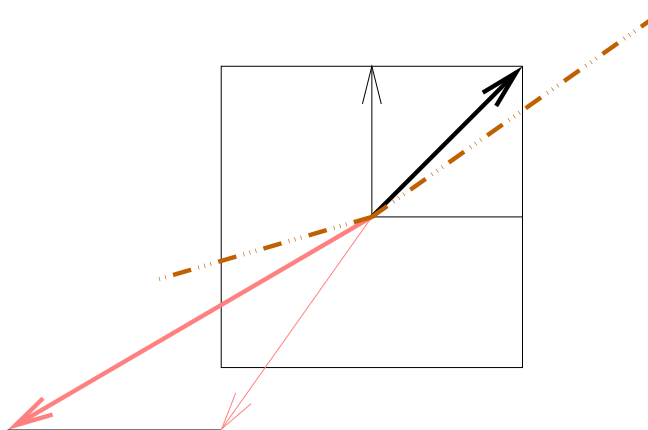


new spiral, Choice 1

$v=1$   
observer  $m_3$



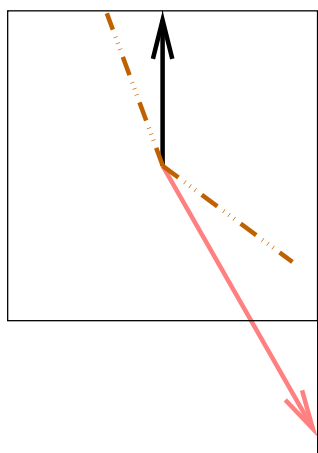
Godel dervish, Choice 2



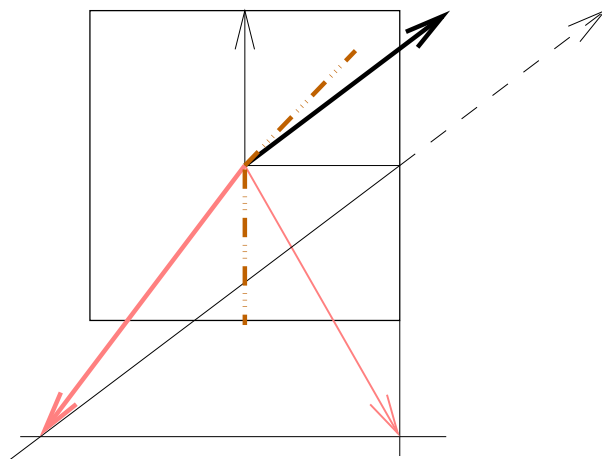
new spiral, Choice 2

Figure 65: Details of observer  $m_3$ .





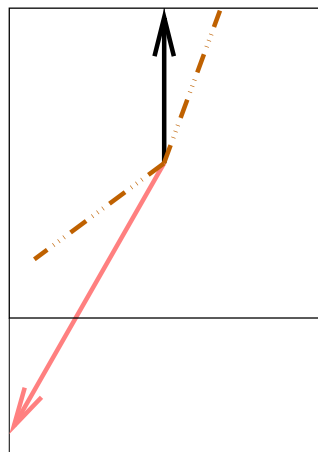
Godel dervish, Choice 1



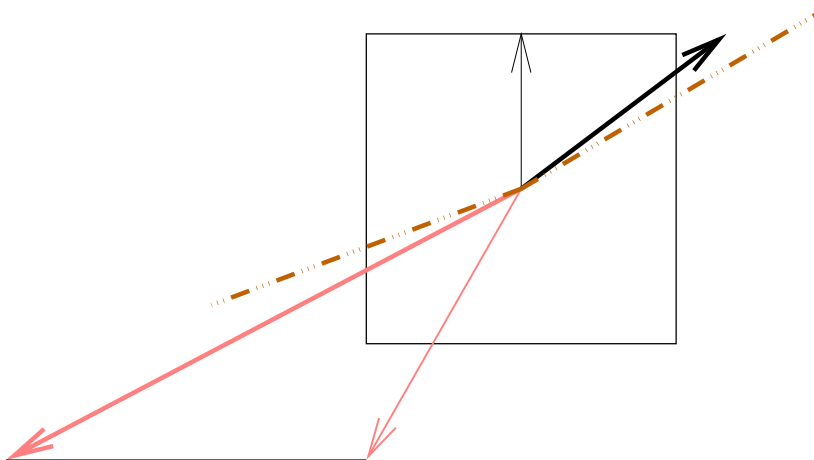
new spiral, Choice 1

$v=1.325$

observer  $m_4$

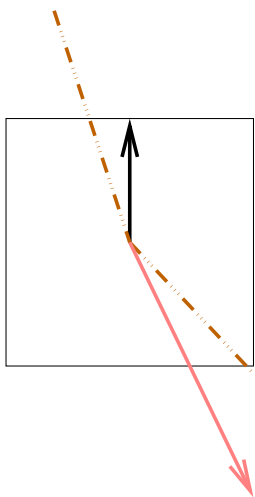


Godel dervish, Choice 2

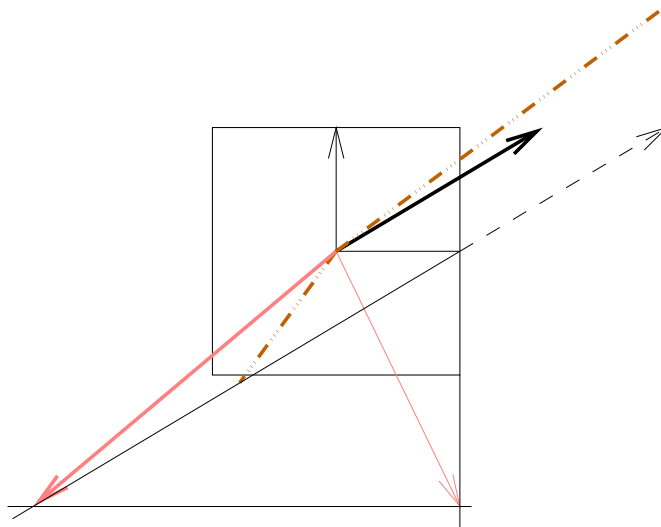


new spiral, Choice 2

Figure 66: Details of observer  $m_4$ .



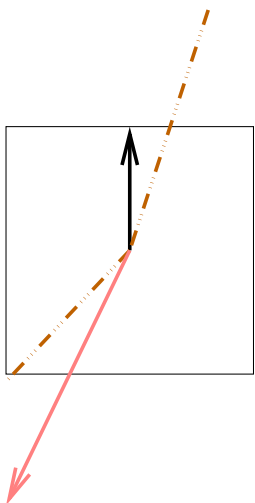
Godel dervish, Choice 1



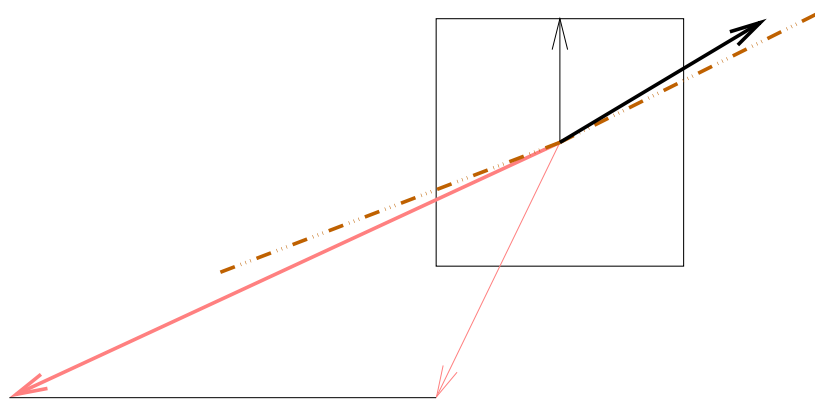
new spiral, Choice 1

$v=1.672222$

observer  $m_5$

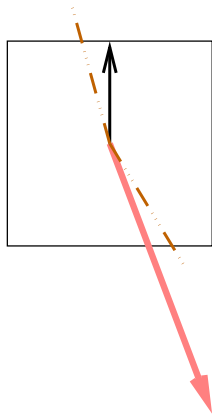


Godel dervish, Choice 2

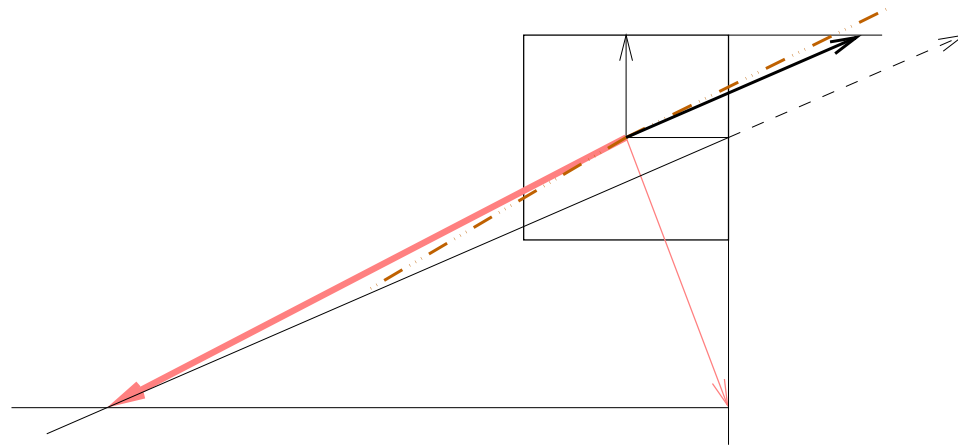


new spiral, Choice 2

Figure 67: Details for observer  $m_5$ .

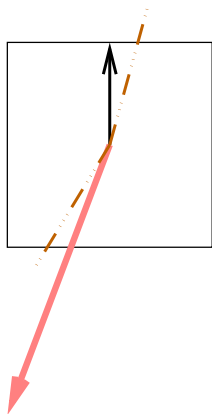


Godel dervish, Choice 1

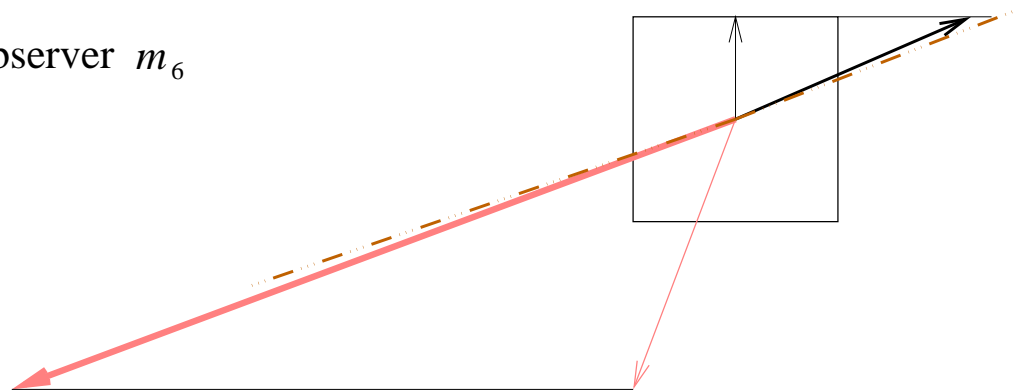


new spiral, Choice 1

$v=2.3008333$   
observer  $m_6$



Godel dervish, Choice 2



new spiral, Choice 2

Figure 68: Details for observer  $m_6$ .

Figure 69: Map 1

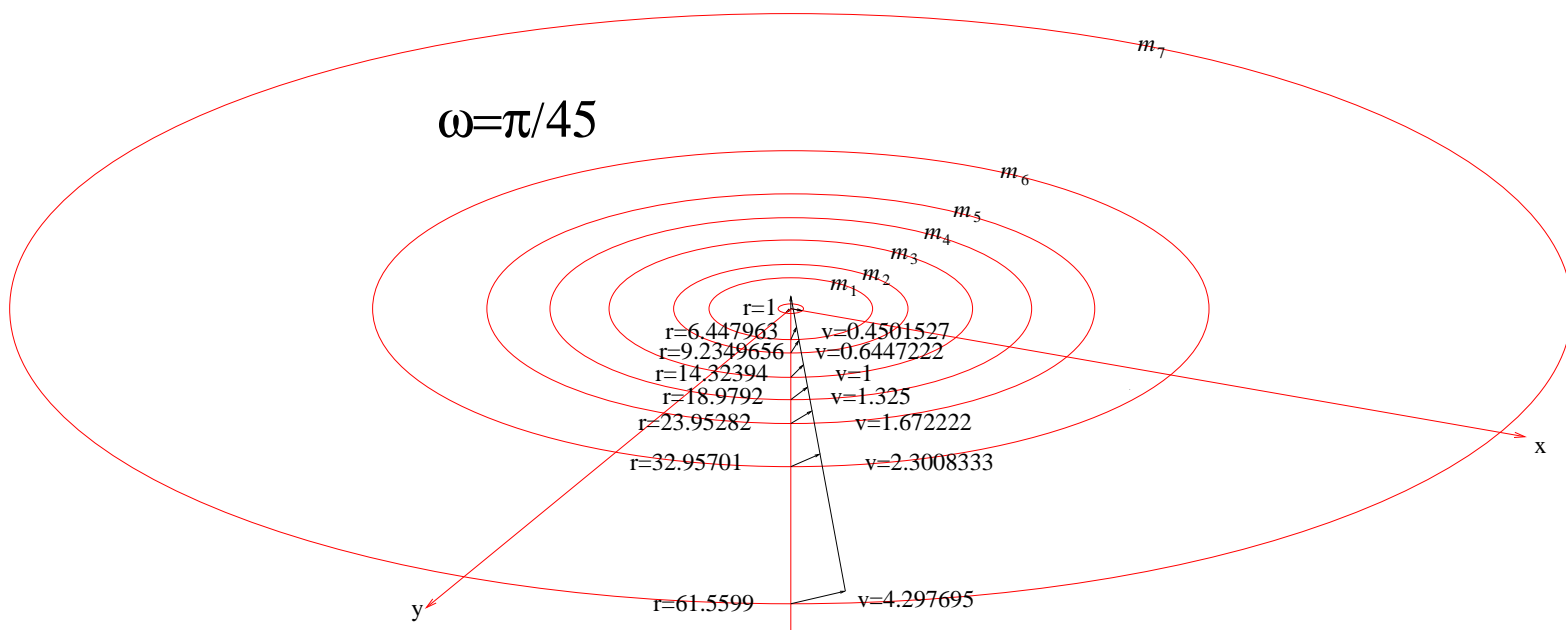
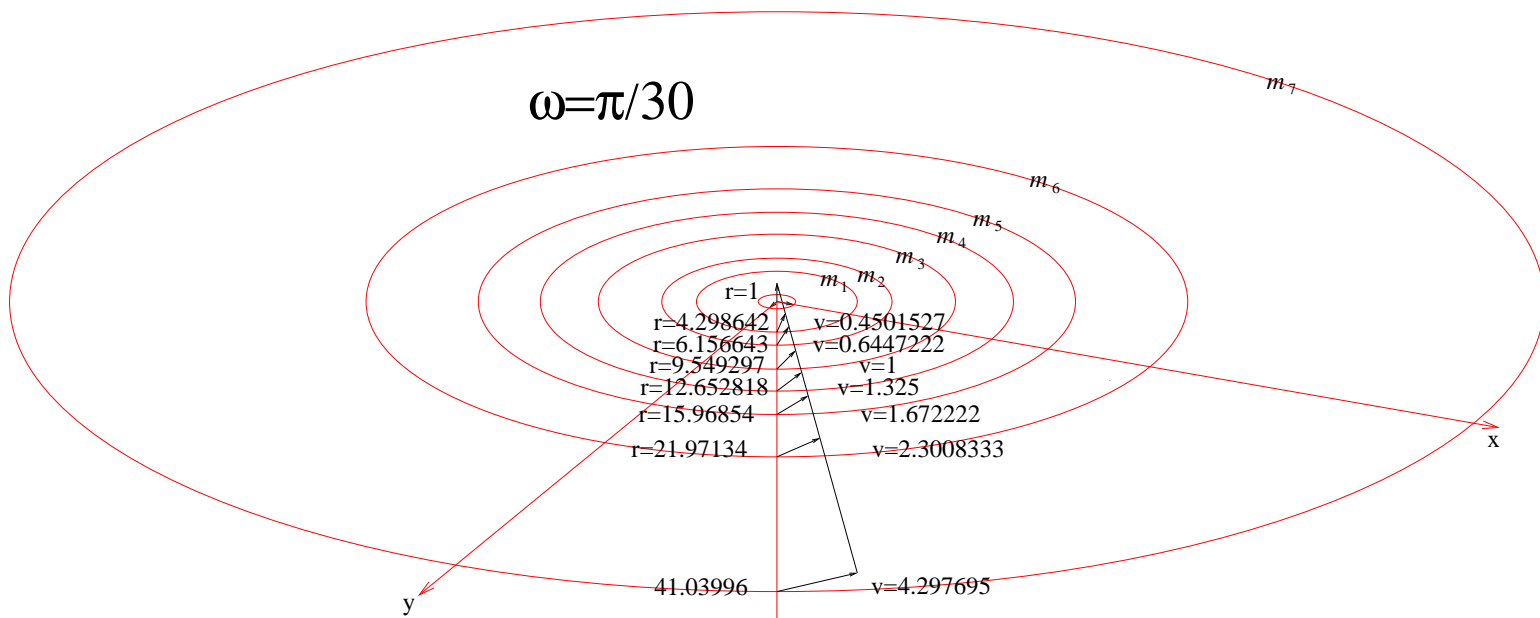


Figure 70: Map 2



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